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NOTETAKER CHECKLIST FORM

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Name: Ian Coley **Email/Phone:** msri@iancoley.org

Speaker's Name: Georg Tamme

Talk Title: On the K-theory of pullbacks

Date: 3 / 27 / 19 **Time:** 11 : 30 am / pm (circle one)

Please summarize the lecture in 5 or fewer sentences:

Milnor squares of rings have some nice properties, including a long exact sequence that unfortunately stops at K_1.

To extend it beyond K_1, they pass to E_1-ring-spectra and enhance the theory thereby.

CHECK LIST

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ON THE K-THEORY OF PULLBACKS

GEORG TAMME

Joint with M. Land.

Definition 1. A square of (possibly commutative) rings (we'll denote $\square$) of the form

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A' & \longrightarrow & B' \end{array}$$

is called a *Milnor square* if it is a pullback square and the vertical maps are surjective.

Theorem 2 (Excision; Bass, Milnor, Murthy '60s)**.** If $\square$ is a Milnor square, then there exists a long exact sequence on K-groups

$$K_1(A) \to K_1(A') \oplus K_1(B) \to K_1(B') \xrightarrow{\partial} K_0(A) \to \cdots$$

continuing infinitely to the right.

But it does not continue to the left – can we extend it?

Theorem 3 (Swan '71)**.** There is no functor $\widetilde{K}_2\colon \text{Rings} \to \mathbf{Ab}$ that would extend that sequence to the left.

It turns out that we can go to the left, but we have to change B'. To do so, we allow arbitrary $\mathbb{E}_1$-rings. Then a Milnor square of $\mathbb{E}_1$-rings will just be a pullback square.

Let $\mathbf{Cat}^{\text{perf}}$ be the category of small, idempotent complete stable ∞-categories. $\text{Perf}(A)$ is in this category for any $\mathbb{E}_1$-ring A.

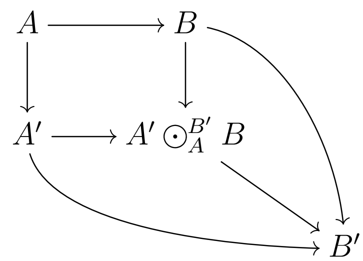
Definition 4. (1) A sequence $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ in $\mathbf{Cat}^{\text{perf}}$ is *exact* if $\mathcal{A} \to \mathcal{B}$ is fully faithful, the composite is zero, and $\mathcal{B}/\mathcal{A} \to \mathcal{C}$ is an equivalence.
(2) A functor $E\colon \mathbf{Cat}^{\text{perf}} \to \text{Sp}$ is a *localizing invariant* if it sends exact sequences to (co)fiber sequences.

Notes by Ian Coley.

Example 5. Nonconnective K-theory, THH, TC.

Now, fix k a connective $\mathbb{E}_\infty$-ring for our base, so we will work with $\mathbb{E}_1$-k-algebras instead.

Theorem 6 (L-T)**.** Any pullback square $\square$ in $\mathbb{E}_1$-k-algebras gives rise to a new commutative square



such that any localizing invariant sends the inside square to a pullback square in Sp and the underlying spectrum of $A' \odot_A^{B'} B$ is $A' \otimes_A^{\mathbb{L}} B$.

Example 7. If $\square$ is a Milnor square, we can boost it up to a pullback square of discrete $\mathbb{E}_1$-rings. Then

$$\pi_i(A' \odot_A^{B'} B) = \mathrm{Tor}_A^i(A', B)$$

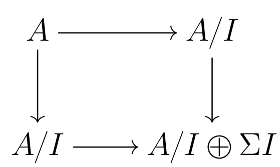
so, in particular, $\pi_0(A' \odot_A^{B'} B) = B'$.

Thus what we're getting is some thickening of our original picture.

Definition 8. A localizing invariant E is *truncating* if $E(A) \to E(\pi_0 A)$ i an equivalence for any connective $\mathbb{E}_1$-k-algebra A.

Corollary 9. Any truncating invariant D satisfies excision and nil-invariance, i.e. $E(A) \to E(A/I)$ is an equivalence for any discrete ring A and nil-ideal $I \subset A$.

Proof. For excision, that $\pi_0(A' \odot_A^{B'} B) \to \pi_0(B')$ is an equivalence is enough (following the reasoning along). For nil-invariance, we may assume that $I \subset A$ is square-zero (by induction). Then we have the following pullback square in $\mathbb{E}_1$-rings:



but $\pi_0(A/I \odot_A^{A/I \oplus \Sigma I} A/I) = A/I$ so applying E to that diagram and using truncating gives us enough equivalences to prove nil-invariance. $\square$

Example 10. Let's now look at examples of truncating invariants, along with the original folks who proved the corollary in the specific cases:

- $HP(-/\mathbb{Q})$ on $\mathbb{Q}$-algebras, proven truncating by Goodwillie. Corollary proven by Cuntz-Quillen.
- $K_{\mathbb{Q}}^{\text{inf}} = \text{fib}(K_{\mathbb{Q}} \to HN(-\otimes\mathbb{Q}/\mathbb{Q})$, proven truncating by Goodwillie. Corollary proven by Cortiñas.
- $K^{\text{inv}} = \text{fib}(K \to TC)$, proven truncating by Dundas-Goodwillie-McCarthy. Corollary proved by Geisser-Hesselholt and Dundas-Kittang.
- $L_{K(1)}K$ on $\mathbb{Z}/N$-algebras for $N > 0$, proven truncating by L-T and Meier. As an application, using the corollary we have $L_{K(1)}K(\mathbb{Z}/p^n) \simeq L_{K(1)}K(\mathbb{Z}/p)$, where p is the prime implicit in the $K(1)$-localization. Quillen showed that $L_{K(1)}K(\mathbb{Z}/p) \simeq 0$.

Now, K-theory is not truncating, but:

Lemma 11 (Waldhausen)**.** If $C \to C'$ is an n-connective map of connective $\mathbb{E}_1$-rings $(n \geqslant 1)$, then $K(C) \to K(C')$ is $(n+1)$-connective.

Corollary 12. If our square $\square$ is a Milnor square and $\text{Tor}^i_A(A', B) = 0$ for $i = 1, \ldots, n-1$, then $A' \odot_A^{B'} \to B'$ is n-connective and thus

$$\begin{array}{ccc} K(A) & \longrightarrow & K(B) \\ \downarrow & & \downarrow \\ K(A') & \longrightarrow & K(B') \end{array}$$

is n-cartesian (basically the definition). So concretely, the original long exact sequence can be extended backwards to K_n.

Remark 13. This implies the result of Suslin on excision, which depended on a slightly stronger vanishing condition on Tor.

Proof. (of the main theorem) Considering a span $\mathcal{A}' \xrightarrow{p} \mathcal{B}' \xleftarrow{q} \mathcal{B}$ in $\mathbf{Cat}^{\text{perf}}$, we can take its lax pullback (aka comma category). *Editorial remark:* Tamme denotes this category $\mathcal{A}' \overrightarrow{\times}_{\mathcal{B}'} \mathcal{B}$ but it is more commonly denoted (p/q) or $p \downarrow q$ in higher category theory, and we will use (p/q) for brevity.

Concretely, its objects are $a' \in \mathcal{A}', b \in \mathcal{B}$ with a map $f\colon p(a') \to q(b)$ in $\mathcal{B}'$. Its morphisms are induced by those in $\mathcal{A}'$ and $\mathcal{B}$.

Lemma 14. There is a split exact sequence

$$\mathcal{B} \to (p/q) \to \mathcal{A}'$$

where this first map is the inclusion $b \mapsto (0, b, 0)$ and the second is the projection. The splittings are given by the projection and the inclusion $a' \mapsto (a', 0, 0)$ respectively.

Thus for any localizing invariant, $E(p/q) \simeq E(\mathcal{A}') \oplus E(\mathcal{B})$ in a canonical way.

Now, consider our original square $\square$ in $\mathbb{E}_1$-rings. Applying $\operatorname{Perf}(-)$ we end up in $\mathbf{Cat}^{\mathrm{perf}}$ and we get a map $i\colon \operatorname{Perf}(A) \to (\operatorname{Perf}(p)/\operatorname{Perf}(q))$. We have that i is fully faithful if and only if $\square$ is a pullback.

Now, let $Q = \operatorname{cof}(i)$, concretely the idempotent completion of the Verdier quotient. This we get a cartesian square in spectra after applying some localizaing invariant E, where we will write $E(C)$ for $E(\operatorname{Perf}(C))$:

$$\begin{array}{ccc} E(A) & \longrightarrow & E(B) \\ \downarrow & & \downarrow \\ E(A') & \longrightarrow & E(Q) \end{array}$$

The claim is that $Q = \operatorname{Perf}(\text{something})$ that we can actually write down. Well, as Q is the quotient of the comma category $(\operatorname{Perf}(p), \operatorname{Perf}(q))$, it is generated by the images of the generators of $\operatorname{Perf}(A')$ and $\operatorname{Perf}(B)$, namely $(A', 0, 0)$ and $(0, B, 0)$. Call these elements $\overline{A}', \overline{B} \in Q$.

Claim: $\overline{B}$ generated Q. Actually, either one does, but it's easier to see it this way. There is a fibre sequence in Q of the form

$$(0, B, 0) \to i(A) = (A', B, \mathrm{can}) \to (A', 0, 0)$$

so in Q we have that $\overline{B}$ is a shift of $\overline{A'}$ so we only need one of them to generate the whole thing. Using Schwede-Shipley recognition principle, we have that $Q \simeq \operatorname{Perf}(\operatorname{End}_Q(\overline{B}))$. Thus we declare $A' \odot_A^{B'} B := \operatorname{End}_Q(\overline{B})$ as an $\mathbb{E}_1$-ring.

The final computation that π_0 is just B' comes from passing to Ind-completions and computing using the big categories. $\square$

Example 15. Let k be any discrete ring, then we have a cartesian square in rings (and $\mathbb{E}_1$-rings)

$$\begin{array}{ccc} k & \longrightarrow & k[t^{-1}] \\ \downarrow & & \downarrow \\ k[t] & \longrightarrow & k[t,t^{-1}] \end{array}$$

where all maps are inclusions.

Proposition 16. $k[t] \odot_k^{k[t,t^{-1}]} k[t^{-1}] = k\langle x,y\rangle/(yx=1)$, the noncommutative polynomial algebra with the given relation. This is also known as the Toeplitz ring over k, $\mathcal{T}_k$.

As an application, there is an exact sequence of nonunital rings

$$0 \to M(k) \to \mathcal{T}_k \to k[t,t^{-1}] \to 0$$

where $M(k)$ is the colimit of $M_n(k)$. Applying any localizing invariant, we get that $E(M(k) \to \mathcal{T}_k)$ is nulhomotopic. Applying the machinery above and Morita invariance, we get a fibre sequence

$$E(M(k)) \simeq E(k) \xrightarrow{0} E(\mathcal{T}_k) \to E(k[t,t^{-1}])$$

and by the main theorems, this allows us to identify

$$E(\mathcal{T}_k) \simeq E(k) \oplus NE(k) \oplus NE(k)$$

where $NE(k) = \mathrm{cof}(E(k) \to E(k[t,t^{-1}])$. Then

$$E(k[t,t^{-1}]) \simeq E(k) \oplus NE(k) \oplus NE(k) \oplus \Sigma E(k)$$

so it's some other form of the Fundamental Theorem of K-theory.

Example 17. We have another cartesian square:

$$\begin{array}{ccc} k[x,y]/(xy) & \longrightarrow & k[y] \\ \downarrow & & \downarrow \\ k[x] & \longrightarrow & k \end{array}$$

where the maps are the quotients you think they are, giving $k[x] \odot^k_{k[x,y]/(xy)} k[y] \simeq k[t]$ with $|t| = 2$.

Example 18. The cocartesian square in spectra yields a cartesian one when applying $\mathrm{Map}(-, k)$:

$$\begin{array}{ccc} S^0 & \longrightarrow & * \\ \big\downarrow & & \big\downarrow \\ * & \longrightarrow & S^1 \end{array} \quad \mapsto \quad \begin{array}{ccc} k^{S^1} & \longrightarrow & k \\ \big\downarrow & & \big\downarrow \\ k & \longrightarrow & k \times k \end{array}$$

and in this case, $k \odot_{k^{S^1}}^{k \times k} k \simeq k[t]$ (the usual one). This is helpful for computing $K(k^{S^1})$.