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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Jeffrey Heninger **Email/Phone:** jeffrey.heninger@yahoo.com

Speaker's Name: Richard Montgomery

Talk Title: Infinitely Many Coplanarities

Date: 11 / 26 / 2018 **Time:** 9 : 30 **am / pm (circle one)**

Please summarize the lecture in 5 or fewer sentences: Consider the four body problem in space with zero angular momentum. If the motion is bounded for sufficiently long, then the masses must go coplanar. If the motion is bounded for all time, then the masses must go coplanar infinitely many times.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☑ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☑ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
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  - **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
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- ☑ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
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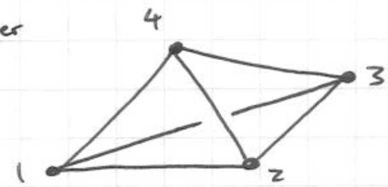
Infinitely Many Coplanarities

Richard Montgomery

Notes by Jeffrey Heninger



4-Body Problem in space (point masses)
think of it as a moving symplex

$q_a(t) \in \mathbb{R}^3$, $a = 1, 2, 3, 4$, $m_a > 0$

$$m_a \ddot{q}_a = \sum_{b \neq a} F_{ab} \quad , \quad F_{ab} = -G \frac{m_a m_b}{r_{ab}^2} \frac{(q_a - q_b)}{r_{ab}} \quad , \quad r_{ab} = |q_a - q_b| \qquad \textbf{(N)}$$

Configuration Space $\quad q = [q_1^{\downarrow}\ q_2^{\downarrow}\ q_3^{\downarrow}\ q_4^{\downarrow}] \in M(3,4) = M.$

<u>Def</u> q is "degenerate" or "coplanar" if the q_a lie in a single plane.

$\iff$ vol($\triangle$ $q_1 q_2 q_3 q_4$) $= 0$

Locus of all degenerate configurations $\Sigma \subset M$

Conservation laws:

Angular Momentum	$J = \sum m_a q_a \times \dot{q}_a$	Center of Mass Frame: wlog,
Linear Momentum	$p = \sum m_a \dot{q}_a$	$\sum m_a \dot{q}_a = 0$
Energy		$\sum m_a q_a = 0$

<u>Def</u> Solution to (N) is called bounded (bdd) if $\exists c > 0$:

$r_{ab}(t) \leq c$, $\quad t \in I \subset \mathbb{R}$

(B) $\qquad\qquad$ ↑ some time interval

"<u>Thm</u>" (imprecise statement): Σ is a global slice for $J=0$, bdd solutions.
every $J=0$, bdd solution has to go coplanar over and over again

$M = \sum m_a$; $\omega^2 = \frac{GM}{c^3}$; ω has units $\frac{1}{\text{Time}}$

<u>Main Thm</u> [M-2018]

If $q : I \to M$ solves (N), has $J = 0$, and satisfies (B),
then q goes coplanar provided $|I| \geq \pi/\omega$

$\exists t_* \in I$ with $q(t_*) \in \Sigma$

<u>Cor</u> If $I = [0, \infty)$, the solution goes coplanar infinitly often.

Inspiration:

I. $N=3$ in $\mathbb{R}^2$ [2002]

same hypotheses ($J=0$, bdd)

then infinitely many syzygy / collinearity

II. Littlejohn

shape space for this problem is $\mathbb{R}^6$

so 3-body ~~solution~~ result can be generalized

Set-up

Mass inner product on configuration space. $\langle\, , \rangle = \langle\, , \rangle_m$ on M

$$K = \frac{1}{2}\sum m_a \dot{q}_a \cdot \dot{q}_a = \frac{1}{2}\langle \dot{q}, \dot{q}\rangle_m$$

$$\langle v, w\rangle = \sum m_a v_a \cdot w_a$$ usual dot product in $\mathbb{R}^3$

Potential Energy $V: M \to \mathbb{R}$ $\qquad V = -G \sum_{a<b} \frac{m_a m_b}{r_{ab}}$

Energy ~~H~~ $= K + V$

(N) $\ddot{q} = -\nabla V(q)$

gradient defined using mass inner product $dV(q)(w) = \langle \nabla V(q), w\rangle_m$

Symmetry group $SE(3)$

(translations) $q_a \mapsto q_a + c$, $c \in \mathbb{R}^3$

(rotations) $q_a \mapsto Rq_a$, $R \in SO(3)$

upstairs $\quad M(3,4)$

$\downarrow$ quotient by $\mathbb{R}^3$ $\qquad$ Metric submersion.

$M(3,3)$ $\leftarrow$ center of mass $= 0$ $\qquad$ Push (N) down.

$\downarrow$ quotient by $SO(3)$

downstairs $\quad Sh(3,4)$ $\quad (\simeq \mathbb{R}^6)$ $\leftarrow$ shape space $\qquad$ Σ is a subset of each

tetrahedra up to rigid motions

on M, $\ddot{q} = -\nabla V$

on Sh, $\nabla_{\dot\gamma} \dot{\gamma} = -\nabla V$ only if $J=0$.

Metric on Shape Space

$$d_{Sh}(\sigma_1, \sigma_2) = d_M(\pi^{-1}(\sigma_1), \pi^{-1}(\sigma_2))$$

distance between inverse images (which is a fiber) ~~=~~ upstairs

comes from a Riemannian metric on Sh, Levi-Civita connection (Riemannian submersion)

except - problems if all masses are colinear

Σ is totally geodesic downstairs, but not upstairs

$\Sigma \subset M(3,3)$ satisfies $\Sigma = \{\det(q) = 0\}$

$\mathcal{L} \subset \Sigma$ collinear
where projection is bad

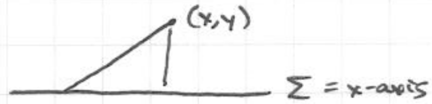
$S(q)$ = signed dist (q, Σ)

why signed? pretend $M = \mathbb{R}^2$:

$\text{dist}((x,y), \Sigma) = |y|$

signed dist $= y$

this is easier to differentiate



In Sh_y, Σ divides tetrahedra with vol > 0 from vol < 0.

Solution $q(t)$ with $J = 0$ $\qquad S(t) = S(q(t))$

S has some singularities that we want to avoid. ← ~~Have~~ Has to deal with double eigenvalues for singular value decomposition of q.

$\text{Sing}(S) = \{q : S \text{ not smooth around } q\}$

<u>Thm</u> A. If q avoids $\text{Sing}(S)$, then

$$\ddot{S} = -Sg, \quad g > 0 \quad \text{and } g \geq \omega^2 \text{ if (B) holds}$$

<u>Thm</u> B. $\text{codim}(\text{Sing}(S)) = 2$

Use A - use Sturm-Liouville to compare to harmonic oscillator

Use B - control $\text{Sing}(S)$

<u>Proof of Thm A</u>

S is invariant under our group $G = SE(3)$ $\qquad S(q(t))$.

Compute derivatives.

$$\dot{S} = \langle \nabla S(\gamma(t)), \underset{\uparrow \text{ downstairs}}{\dot{\gamma}} \rangle \qquad \gamma = \pi(q)$$

$$\ddot{S} = \langle \nabla S, \nabla_{\dot\gamma} \dot\gamma \rangle + \langle \nabla_{\dot\gamma} \nabla S, \dot\gamma \rangle$$

To show

$$= \langle \nabla S, -\nabla V \rangle + Q(\dot\gamma, \dot\gamma)$$

$$= -\underbrace{S' g_1}_{\text{I}} - \underbrace{S g_2}_{\text{II}}$$

Thm A $\Leftrightarrow$ I: $g_1 \geq 0$, $g_1 \geq \omega^2$ if (B)

II: $g_2 = g_2(\cdot,\cdot) \geq 0$

A4

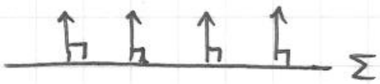
Proof of I: Key $\|\nabla S\| = 1$

(~~Hamiltonian~~ Hamilton-Jacobi if $V \equiv 0$ $\quad H(q, dS(q)) = \frac{1}{2}|dS(q)|^2$
$= \frac{1}{2}|\nabla S(q)|^2 = c$)

True for signed distances from any Σ
in any Riemannian manifold.

$S = y$, $\nabla S = e_2$, $\Sigma_c = \{S = c\}$

helpful to think of 2-d



$\boxed{\tau = S(q(\tau))}$

integral curves of ∇S are geodesics perpendicular to Σ

Solve $\frac{dq}{d\tau} = \nabla S(q)$, $q_0 \in \Sigma$

upstairs, space is flat,
so geodesics are trivial

$q(\tau) = q_0 + \tau v$, $v \perp \Sigma$

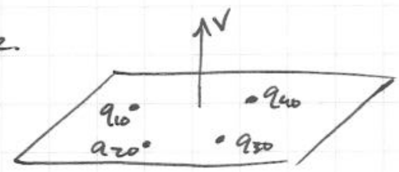
$\frac{d}{d\tau}(-V(q(\tau))) = \langle \nabla S, -\nabla V \rangle$

$$r_{ab}(\tau)^2 = |(q_a + \tau v_a) - (q_b + \tau v_b)|^2$$
$$= |q_{ab}|^2 + 2\tau \underbrace{r_{ab} \cdot q_{ab}}_{=0} + \tau^2 |v_{ab}|^2$$

why? ✪

$q_{ab} = q_a - q_b$, $v_{ab} = v_a - v_b$

✪ $q_0 \in \Sigma$, so all q_{a0}'s are on a plane.
Say $\mathbb{R}^2 \subset \mathbb{R}^3$.
Each $v_a \in \mathbb{R} e_3$ with $e_3 \perp \mathbb{R}^2$



$$\frac{d}{d\tau} r_{ab}^2 = 2\tau |v_{ab}|^2 = 2S|v_{ab}|^2$$

$$\frac{d}{d\tau}\left(G \sum \frac{m_a m_b}{r_{ab}}\right) = -S \underbrace{G \sum \frac{m_a m_b |v_{ab}|^2}{r_{ab}^3}}_{g_1}$$

$$|\nabla S|^2 = \sum m_a v_a \cdot v_a = \frac{\sum m_a m_b |v_{ab}|^2}{\sum m_a} = I$$

$\Rightarrow$ (B) $\Rightarrow g_1 \geq \omega^2$