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Name: Tony Feng **Email/Phone:** tonyfeng@stanford.edu

Speaker's Name: Janos Kollar

Talk Title: Moduli of canonical models

Date: 2 / 4 / 19 **Time:** 10 : 30 am / pm (circle one)

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Moduli spaces of algebraic varieties III

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Theorem Let $X \to S$ be a flat, projective morphisms with stable fibers, S reduced. Equivalent:

1. The volume of the fibers $s \mapsto (K_{X_s}^n)$ is locally constant.
2. The plurigenera $s \mapsto h^0(X_s, \omega_{X_s}^{[m]})$ are locally constant.
3. $\omega_{X/S}^{[m]}$ is flat and commutes with base change $\forall m$.

(3) $\Rightarrow$ (2)

(3): $\omega_{X/S}^{[m]}$ is flat and commutes with base change $\forall m$.

(3') the $\omega_{X_s}^{[m]}$ form a flat family,

(3") $\chi(X_s, \omega_{X_s}^{[m]})$ is locally constant.

$m \geq 2$: we expect that $H^i(X_s, \omega_{X_s}^{[m]}) = 0$ for $i > 0$.
Note: K-V not enough need Ambro-Fujino vanishing

$m = 1$: we expect $H^i(X_s, \omega_{X_s}) = H^{n-i}(X_s, \mathcal{O}_{X_s})$ and

K-Kovács: X_s Du Bois, so $H^{n-i}(X_s, \mathcal{O}_{X_s})$ locally constant.

(Duality fails since X_s need not be CM, but ok.)

Typical example (with divisors)

$X = (xy - uv = 0)$, $u - v : X \to \mathbb{C}^1_t$,
$D := (x = u = 0) + (y = v = 0)$.

– $t \neq 0$: X_t smooth, D_t Cartier and $(D_t^2) = 0$.
– $t = 0$: X_0 cone , D_0 Cartier and $(D_0^2) = 2$.

ideal of D is $(xy, xv, uy, uv) \to (xv, uy, uv)$.
Central fiber: $(xu, uy, u^2) \subset (u)$, embedded point at origin.

Typical situation (with sheaves)

F reflexive sheaf on X, locally free in codimension 1 on all fibers. We get

$$r : F \twoheadrightarrow F|_{X_s} \hookrightarrow \left(F|_{X_s}\right)^{**}.$$

Corollary:

- $s \mapsto H^0\left(X_s, \left(F|_{X_s}\right)^{**}\right)$ is upper semicontinuous
- If the $\left(F|_{X_s}\right)^{**}$ are globally generated then locally constant $\Leftrightarrow$ $F|_{X_s}$ is reflexive.

(2) $\Rightarrow$ (3)

(2): $s \mapsto h^0(X_s, \omega_{X_s}^{[m]})$ are locally constant.

For $m \gg 1$ the $\omega_{X_s}^{[m]}$ are globally generated, so $\omega_{X/S}^{[m]}$ is flat and commutes with base change for $m \gg 1$.

If $\omega_{X/S}^{[M]}$ is locally free then

$$\omega_{X/S}^{[m+M]} \cong \omega_{X/S}^{[m]} \otimes \omega_{X/S}^{[M]}$$

so $\omega_{X/S}^{[m]}$ is flat and commutes with base change $\forall m$.

$(2) \Rightarrow (1)$

(2): $s \mapsto h^0(X_s, \omega_{X_s}^{[m]})$ are locally constant.

The $(K_{X_s}^n)$ are the leading terms, so
$s \mapsto (K_{X_s}^n)$ are also locally constant.

(1) $\Rightarrow$ (2) slide 1

Theorem Let $X \to S$ be a flat, projective morphisms with S_2 fibers. L reflexive rank 1 sheaf, locally free on codimension 1 on each fiber. Assume that each $(L_s)^{**}$ is locally free and ample. Then

1. $s \mapsto \mathrm{volume}(L_s)^{**}$ is upper semicontinuous, and
2. locally constant iff L is locally free.

Apply this to $\omega_{X/S}^{[M]}$ such that every $\omega_{X_s}^{[M]}$ is locally free. Need extra work for the other m.

Example

$Y \to \mathbb{C}$ family of degree 4 surfaces,
$Y_0 \supset L$ line, $\rho(Y_t) = 1$.

Contract $L \subset Y$ to get $\pi : Y \to X$ and $X \to \mathbb{C}$.
L is the image of $2H$ on X.

- L_t is ample and $(L_t^2) = 16$.
- $L_0 = \pi_0(2H_0 + L)$ is ample and $(L_0^2) = 18$.
- $X \to \mathbb{C}$ is NOT projective.

(1) $\Rightarrow$ (2) slide 2

$n = 2$ case: here cokernel of $L \to L_0^{**}$ is 0 dimensional, so

$$\chi(X_0, (L_0^{**})^m) \geq \chi(X_g, (L_g^{**})^m) = \chi(X_g, L_g^m).$$

Riemann-Roch:

$$\tfrac{1}{2}(L_0^{**} \cdot L_0^{**})m^2 + b_0 m + \chi(X_0) \geq \tfrac{1}{2}(L_g^{**} \cdot L_g^{**})m^2 + b_g m + \chi(X_g).$$

If $(L_0^{**} \cdot L_0^{**}) = (L_g \cdot L_g)$ then

$$b_0 m \geq b_g m \quad \forall m.$$

So $b_0 = b_g$.

(1) $\Rightarrow$ (2) slide 3

$n \geq 3$ induction (nontrivial) reduces to special case:
L is locally free except at isolated points.

Local Grothendieck-Lefschetz: $x \in X_0 \subset X$ then

$$\mathrm{Pic}(X \setminus \{x\}) \hookrightarrow \mathrm{Pic}(X_0 \setminus \{x\})$$

if $\mathrm{depth}_x X_0 \geq 3$.

Problem: we have only $\mathrm{depth}_x X_0 \geq 2$.

Conjecture. Still ok if $\mathrm{depth}_x X_0 \geq 2$ and $\dim X_0 \geq 3$.

– slc case (K)
– normal case (Bhatt - de Jong)
– general case (K)

Aside: Fulger - K - Lehmann

X normal, proper, D big $\mathbb{R}$ divisor, E effective $\mathbb{R}$ divisor
Equivalent

- $H^0\big(X, \mathcal{O}_X\llcorner m(D-E)\lrcorner\big) = H^0\big(X, \mathcal{O}_X\llcorner mD\lrcorner\big)\ \forall m \geq 1$.
- $\mathrm{volume}(D-E) = \mathrm{volume}(D)$.

Main existence theorem

Theorem

Fix positive n, d. There is a projective coarse moduli space $\bar{M}_{n,d}$ parametrizing stable varieties X of dimension n such that $(K_X^n) = d$.

Main issues:

- existence of 1-parameter limits, irreducible case;
- existence of 1-parameter limits, reducible case;
- boundedness
- moduli of stable pairs

Limits, irreducible case

Original KSB approach, needs MMP
Hacon-McKernan-Xu

Limits, reducible case

What are the limits when the general fiber is reducible?

For curves: normalize, construct limits and then glue together.

Problem: gluing is very hard in higher dimensions.

Example: Glue 2 copies of $\big(\mathbb{P}^2, (xyz = 0)\big)$.
gluing data: $\lambda_x, \lambda_y, \lambda_z \in \mathbb{C}^*$.
When is it projective?

Answer: iff $\lambda_x \lambda_y \lambda_z$ is a root of unity.

Properness, boundedness I

Two possible problems: in a limit we get worse and worse
– sigularities (which mK_S is Cartier?)
– reducible varieties (many components?)

Curve case: $C = \cup_i (C_i, P_i)$ then

$$\deg K_C = \sum_i \big(\deg K_{C_i} + \# P_i\big)$$

So at most $2g(C) - 2$ irreducible components.

Surface case: $S = \cup_i S_i$ then

$$K_S^2 = \sum_i \big(K_{S_i} + D_i\big)^2.$$

Problem: The $\big(K_{S_i} + D_i\big)^2$ are only rational.

Properness, boundedness II

Example: $(\mathbb{P}^2(1,2,3), C \in |\mathcal{O}(7)|)$ then $(K_S + C)^2 = 1/6$.

Example (Alexeev–Liu) $(K_S^2) = \frac{1}{48983}$.

Proposition. For surface pairs $(S, C \neq 0)$ we have

$$(K_S + C)^2 \geq \frac{1}{1764}.$$

Theorem (Alexeev, Hacon–McKernan–Xu) In any dimension

$$\{(K_X + D)^n\} \subset \mathbb{Q}$$

satisfies the descending chain condition.

Effective bounds on surface singularities (Rana, Urzúa)

Stable pairs

Objects: (X, Δ) where

- X is seminormal, proper
- $\Delta = \sum d_i D_i$ effective and $D_i \not\subset \mathrm{Sing} X$

(Mumford divisor)

- $K_X + \Delta$ $\mathbb{Q}$-Cartier

$g : X' \to X$ log resolution then write

$$K_{X'} + \Delta' + \sum a_i E_i \sim g^*(K_X + \Delta)$$

- all $a_i \leq 1$

Stable families

Definition I. $f : (X, \Delta) \to C =$ smooth curve is stable iff

- $K_{X/C} + \Delta$ is $\mathbb{Q}$-Cartier and
- all fibers stable.

Definition II. $f : (X, \Delta) \to S =$ reduced is stable iff

- pull-back to smooth curves is stable.

Definition III. $f : (X, \Delta) \to S =$ arbitrary is stable iff

- ????

Coefficients $> \frac{1}{2}$, slide 1

Theorem. $f : (X, \sum d_i D_i) \to C =$ smooth curve stable and $d_j > \frac{1}{2}$ then $D_j \to C$ is flat with reduced fibers.

Write $dD = d_j D_j$ and $\sum d_i D_i = dD + \Delta$.

Easy case: $n = 1$. Limit case: $\mathbb{C}^2_{xy} \to \mathbb{C}^1_x$, $D = (y^2 = x)$,

Old case: D $\mathbb{Q}$-Cartier. We almost know that X is CM
(Elkik, Alexeev, Fujino, K.)

So $X_c \cap D$ is unmixed. Generically reduced (easy case),
so reduced

Coefficients $> \frac{1}{2}$, slide 2

General case: Equivalent form $X_c \cup D$ is seminormal.

Make D $\mathbb{Q}$-Cartier:
get $g : (X', X'_0 + dD' + \Delta') \to (X, X_0 + D + \Delta)$
such that $-D'$ is g-nef.

$$0 \to \mathcal{O}_{X'}(-X'_0 - D') \to \mathcal{O}_{X'} \to \mathcal{O}_{X'_0 + D'} \to 0$$

$$g_* \mathcal{O}_{X'} = \mathcal{O}_X \to g_* \mathcal{O}_{X'_0 + D'} \to R^1 g_* \mathcal{O}_{X'}(-X'_0 - D').$$

Note that

$$-X'_0 - D' \sim K_{X'} + \Delta' + (1 - d)(-D').$$

G-R should give that $R^1 g_* \mathcal{O}_{X'}(-X'_0 - D')$ is zero.
Not good enough.

Coefficients $> \frac{1}{2}$, slide 3

Ambro-Fujino implies:
$R^1 g_* \mathcal{O}_{X'}(-X_0' - D')$ has no associated primes contained in $\mathrm{Supp}(X_0 + D)$.

So $\mathcal{O}_X \to \mathcal{O}_{X_0+D} \to g_* \mathcal{O}_{X_0'+D'}$ is onto so
$\mathcal{O}_{X_0+D} = g_* \mathcal{O}_{X_0'+D'}$.

Lemma: $g : Y' \to Y$ proper, birational, $g_* \mathcal{O}_{Y'} = \mathcal{O}_Y$.
Then Y' seminormal $\Rightarrow$ Y seminormal.