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Talk Title: Infinity categories and why they are useful, II

Date: 2 / 4 / 19 **Time:** 9 : 00 am / pm (circle one)

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INFINITY CATEGORIES AND WHY THEY ARE USEFUL, II

CARLOS SIMPSON

1. Fibrations and fundamental groupoids

Today's slogan is: "a fibration $Y \to X$ of spaces is the same thing as an action of the *Poincaré* ∞*-groupoid* $\pi_\infty(X)$".

To explain this, we recall that a covering space $Y \to X$ is the same thing as an action of the "Poincaré 1-groupoid" $\pi_1(X)$, i.e. a functor $\pi_1(X) \to \mathrm{Sets}$. This groupoid $\pi_1(X)$ has objects being the (discrete) set of points $x \in X$, and morphisms between x and y being homotopy classes of paths $x \rightsquigarrow y$.

In this equivalence, a covering $f \colon Y \to X$ corresponds to the functor $F \colon \pi_1(X) \to \mathrm{Sets}$ taking $x \in X \mapsto Y_x := f^{-1}(x)$.

The Poincaré ∞-groupoid $\pi_\infty(X)$ is supposed to be a "spatially enriched category", whose objects are the points of X and $\mathrm{Map}(x, y)$ is the path space $P(x, y)$ between x and y.

How do you organize this into a nice structure? We have a composition law for paths, but the concatenation of paths is not associative "on the nose".

If X is connected, all the objects of $\pi_\infty(X)$ will be equivalent. This category is then basically the same thing as $P(x, x)$, considered as a kind of "group object". This is the loop space $\Omega_x(X)$.

2. Delooping machines

What is the extra structure on $\Omega_x(X)$ needed to recover X? This question has been studied by topologists, who have introduced various "delooping machines", which are ways of talking about the structure.

The first step is an H-space, which is a group object in the homotopy category of spaces. This is not enough.

Segal gave a very elegant solution. We can define $\mathcal{P} = \mathcal{P}_X$ to be the *loop groupoid* of X, as follows: for any $n \geq 0$ let

$$\Delta^n = \{(t_0, \ldots, t_n) \in \mathbf{R}^{n+1} \mid t_i \geq 0, \sum t_i = 1\}$$

be the standard n-simplex.

Let $\mathcal{P}_n$ be the space of maps $\Delta^n \to X$ discretized on the vertices:

$$\mathcal{P}_n = \mathrm{Map}(\Delta^n, X) \times_{\mathrm{Map}(\{0,\ldots,n\}, X)} X^{n+1}.$$

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In other words,

$$\mathcal{P}_n = \coprod_{x_0,\ldots,x_n} \mathcal{P}_n(x_0,\ldots,x_1) = \mathrm{Maps}^{x_0,\ldots,x_n}(\Delta^n, X).$$

Example 2.1. For $n = 1$, we have

$$\mathcal{P}_1 = \coprod_{x,y\in X} \mathcal{P}(x,y).$$

This is the path space

$$\mathcal{P}(x,y) = \mathrm{Map}_{0\to x, 1\to y}([0,1], X) = \mathrm{Paths}_X(x,y).$$

These are naturally organized into a simplicial space

$$\mathcal{P}_1 \rightrightarrows \mathcal{P}_0$$

Note that $\mathcal{P}_0$ is a discrete set.

This simplicial space $\mathcal{P}_\bullet$ has the following nice properties:

(a) the map

$$\mathcal{P}_n \to \mathcal{P}_1 \times_{\mathcal{P}_0} \mathcal{P}_1 \times_{\mathcal{P}_0} \ldots \times_{\mathcal{P}_0} \mathcal{P}_1$$

sending

$$\mathcal{P}_n(x_0,\ldots,x_n) \to \mathcal{P}(x_0,x_1) \times \mathcal{P}(x_1,x_2) \times \ldots \times \mathcal{P}(x_{n-1},x_n)$$

is a weak equivalence. This comes from the fact that Δ^n retracts onto its *spine*, which is the arc connecting $x_0, x_1, \ldots, x_n$. The right hand side is the maps from the spine into X, and the left hand side is the maps from the simplex into X.

Example 2.2. A classical simplicial set satisfies this condition if and only if it is the nerve of a category.

(b) A grouplike condition on $\tau_{\le 1}\mathcal{P}_\bullet$, which is automatically a category (whose nerve is the simplicial set $n \mapsto \pi_0\mathcal{P}_n$), namely that it is a *groupoid*.

Informally, this says that "arrows are invertible up to homotopy".

The structure of a simplicial space $\mathcal{P}_\bullet$ satisfying these condition conditions, and with $\mathcal{P}_0 = \{*\}$, is a delooping machine. Denote by $|\mathcal{P}_\bullet|$ a realization of the simplicial space (i.e. the diagonal realization of the bisimplicial set).

How does the composition show up? We have a diagram

$$\begin{array}{ccc} \mathcal{P}_2 & \xrightarrow{(0,1),(1,2)} & \mathcal{P}_1 \times_{\mathcal{P}_0} \mathcal{P}_1 \\ \downarrow{\scriptstyle (0,2)} & \swarrow & \\ \mathcal{P}_1 & & \end{array}$$

and the condition (a) allows us to "invert" the horizontal arrow and get a multiplication. The analogous diagram for $\mathcal{P}_3$ will give us associativity.

3. Segal categories

Definition 3.1. A *Segal category* is a simplicial space A (i.e. a bisimplicial set $\Delta^{op} \times \Delta^{op} \to \text{Sets}$) such that

(1) A_0 is a discrete set,
(2) For all $n \geq 2$, $A_{n,\bullet} \to A_{1,\bullet} \times_{A_0} A_{1,\bullet} \times_{A_0} \ldots \times_{A_0} A_{1,\bullet}$ (corresponding to $\{i, i+1\} \hookrightarrow [0,n]$) is a weak homotopy equivalence.

A morphism is a morphism of bisimplicial sets.

We sketch how to go back and forth between Segal categories and simplicially enriched categories.

First we explain how to produce a simplicial category from a Segal category. The *truncation* of a Segal category A is the (1-) category $\tau_{\leq 1}(A_\bullet)$ whose nerve is $n \mapsto \pi_0(A_{n,\bullet})$. We have

$$A_1 = \coprod_{x,y \in A_0} A_1(x,y).$$

The objects of A are A_0. The morphisms $\text{Map}_A(x,y) = A_{1,\bullet}(x,y)$. This gives a "simplicially enriched category"-like object.

If B was a simplicial category, put

$$A_n(x_0, \ldots, x_1) = B(x_0, x_1) \times \ldots \times B(x_{n-1}, x_n).$$

The strictly associative composition law of B gives a simplicial set, where the Segal maps are even *isomorphisms*. This gives an embedding from the category of simplicial categories to the category of Segal categories.

Definition 3.2. A morphism $f \colon A \to B$ of Segal categories is a *Dwyer-Kan equivalence* if:

(1) ("fully faithful") for all $x, y \in \text{Ob}(A) = A_0$, the map

$$\text{Map}(x,y) \xrightarrow{f} \text{Map}(fx, fy)$$

is a weak equivalence of spaces.
(2) ("essentially surjective") The functor $\tau_{\leq 1}(A) \to \tau_{\leq 1}(B)$ is essentially surjective, or $\tau_{\leq 0}(A) \to \tau_{\leq 0}(B)$ is surjective.

Theorem 3.3 (Bergner)**.** *The functor from simplicial categories, localized at Dwyer-Kan equivalences, to Segal categories, localized at Dwyer-Kan equivalences, is an equivalence of categories.*

These (simplicial categories and Segal categories) give two models for ∞-categories. There are other models, e.g.

- *quasicategories* (due to Boardman-Vogt, developed by Joyal and Lurie).
- Complete Segal spaces (Rezk).