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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Karol Kozlol **Email/Phone:** kkozlol@ualberta.ca

Speaker's Name: Cong Xue

Talk Title: Finiteness of Cohomology of Moduli of Shtukas

Date: 4 / 9 / 19 **Time:** 11 : 00 am / pm (circle one)

Please summarize the lecture in 5 or fewer sentences: THE SPEAKER DISCUSSED COHOMOLOGY OF THE SHEAVES FROM THE PREVIOUS TALK, AND HOW TO DEFINE THEIR CUSPIDAL PART. SHE THEN SHOWED THAT THIS COHOMOLOGY IS A FINITE TYPE HECKE MODULE

CHECK LIST

(This is **NOT** optional, we will **not pay for incomplete** forms)

☒ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.

☒ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.

- Computer Presentations: Obtain a copy of their presentation
- Overhead: Obtain a copy or use the originals and scan them
- Blackboard: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
- Handouts: Obtain copies of and scan all handouts

☒ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.

☒ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)

☒ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

FINITENESS OF COHOMOLOGY OF MODULI OF SHTUKAS — XUE

§1 INTRO

ASSUME FOR SIMPLICITY: G SEMISIMPLE
w/o LEVEL STRUCTURE

ALREADY DEFINED: $\mathrm{Sht}_{I,W}$, $\mathcal{H}_I(W)$

$$H_I(W) = \mathcal{H}_I(W)\big|_{\overline{\eta}^I}, \quad \overline{\eta}^I \to \eta^I \to X^I \quad \text{GEOM GENERIC PT}$$

V. LAFFORGUE: $H_I(W)^{Hf} := \{ c \in H_I(W) : \dim_E (C_c(G(\mathbb{O}) \backslash G(\mathbb{A}) / G(\mathbb{O}), E) \cdot c) < \infty \}$

"HEIGHT FINITE"

$$\boxed{\text{PROP} \quad H_\emptyset(\mathbb{1})^{Hf} = C_c^{cusp}(G(F)\backslash G(\mathbb{A}) / G(\mathbb{O}), E)}$$

BY DRINFELD LEMMA, $H_I(W)^{Hf} \circlearrowleft \mathrm{Gal}(\overline{F}/F)^I$

EXCURSION OPERATORS: $\forall I, W$, $x \in W$, $\xi \in W^*$, $(\gamma_i) \in \mathrm{Gal}(\overline{F}/F)^I$

$$\begin{tikzcd}
C_c^{cusp} = H_\emptyset(\mathbb{1})^{Hf} \arrow[r, "x"] & \mathcal{H}_I(W)^{Hf}\big|_{\Delta(\overline{\eta})} \arrow[r, "sp^*", "\sim"'] & \mathcal{H}_I(W)^{Hf}\big|_{\overline{\eta}^I} \arrow[d, "(\gamma_i)"] \\
C_c^{cusp} = H_\emptyset(\mathbb{1})^{Hf} & \mathcal{H}_I(W)^{Hf}\big|_{\Delta(\overline{\eta})} \arrow[l, "\xi"'] & \mathcal{H}_I(W)^{Hf}\big|_{\overline{\eta}^I} \arrow[l, "(sp^*)^{-1}"', "\sim"]
\end{tikzcd}$$

$$\Rightarrow \quad C_c^{cusp} = \bigoplus_{\sigma : Gal \to \hat{G}} \mathfrak{h}_\sigma$$

$$\boxed{\begin{array}{l}\underline{\text{THM 1}} \quad \forall v \in |X|, \ H_I(W) \text{ IS OF FINITE TYPE AS AN} \\ \mathcal{H}_v = C_c(G(\mathcal{O}_v)\backslash G(F_v)/G(\mathcal{O}_v), E)\text{-MOD}\end{array}}$$

BY DRINFELD'S LEMMA $H_I(W) \circlearrowleft \text{WEIL}(\bar{F}/F)^I$

$\rightsquigarrow$ CAN EXTEND EXCURSION OPERATORS FROM C_c^{cusp} TO C_c

$$\Rightarrow \quad C_c / I C_c = \bigoplus_{\sigma : Weil \to \hat{G}} \mathfrak{h}_\sigma \quad \text{COMPAT w/ PARABOLIC INDUCTION}$$

$I \subset \mathcal{H}_v$ IDEAL OF FINITE CODIM

$$\boxed{\underline{\text{COR OF THM 1}} \quad H_I(W)^{Hf} \text{ HAS FINITE DIM}}$$

$$\boxed{\underline{\text{THM 2}} \quad \underbrace{H_I(W)^{cusp}}_{\text{TB DEFINED}} = H_I(W)^{Hf}}$$

<u>N.B.</u> $H_I(W)^{Hf}$ IS DENOTED $H_{I,W}$ BY V.L.

$\underline{\S 2}$ $\underline{\text{HARDER – NARASIMHAN FILT'N}}$

$$\begin{array}{ccc} Sht_{I,W} & \supset & Sht_{I,W}^{\leq \mu} \\ \downarrow & \square & \downarrow \\ Bun_G & \underset{\text{open}}{\supset} & Bun_G^{\leq \mu} \end{array} \qquad \mu \text{ DOM'T COWT OF } G$$

<u>EX</u> $G = SL_2$, HAVE $Bun_G^{\leq n}$ (SEE WEINSTEIN'S TALK)

$\mathrm{Sht}_{I,W}^{\leq \mu}$ IS A DM STACK OF FINITE TYPE

$$H_I(W)^{\leq \mu} := H_c^0\left(\left(\mathrm{Sht}_{I,W}^{\leq \mu}\right)_{\overline{\eta}^I}, \mathcal{E}^* \mathrm{SAT}_I(W)\right)$$

F.D. E-V.S.

FOR $\mu_1 \leq \mu_2 \rightsquigarrow \mathrm{Sht}_{I,W}^{\leq \mu_1} \xhookrightarrow[\text{OPEN}]{} \mathrm{Sht}_{I,W}^{\leq \mu_2}$

$$\rightsquigarrow H_I(W)^{\leq \mu_1} \longrightarrow H_I(W)^{\leq \mu_2}$$

$$\boxed{\underline{\text{DEF}} \quad H_I(W) = \varinjlim_{\mu} H_I(W)^{\leq \mu}}$$

$\underline{\S 3}$ $\underline{\text{CONSTANT TERM MORPHISM}}$

$$G \supset P \twoheadrightarrow M$$

$$\begin{array}{ccccc} \mathrm{Sht}_{G,I,\bullet} & \xleftarrow{i} & \mathrm{Sht}_{P,I,\bullet} & \xrightarrow{\pi} & \mathrm{Sht}_{M,I,\bullet} \\ \mathcal{E} \times^P G & \longleftarrow\!\shortmid & \mathcal{E} & \shortmid\!\longrightarrow & \mathcal{E} \times^P M \end{array}$$

(VIEW W AS A REP OF $\widehat{M}$ BY RES)

$$\mathrm{Sht}_{P,I,W} := i^{-1}(\mathrm{Sht}_{G,I,W})$$

$$\mathrm{Sht}_{G,I,W}^{\leq \mu} \longleftarrow \mathrm{Sht}_{P,I,W}^{\leq \mu} \longrightarrow \mathrm{Sht}_{M,I,W}^{\leq \mu}$$

$$\mathrm{Sht}_{P,I,W}^{\leq \mu} = i^{-1}(\mathrm{Sht}_{G,I,W})$$

CAN CONSTRUCT A MORPHISM

$$H_{G,I}(W)^{\leq \mu} \longrightarrow H_{M,I}(W)^{\leq \mu}$$

TAKE $\varinjlim$, GET

$$H_{G,I}(W) \xrightarrow{C_G^P} H_{M,I}(W)$$

$\boxed{\begin{array}{l}\text{PROP } \exists\, c \in \mathbb{Q}_{\geq 0} \text{ ST } \forall \mu \text{ WITH } \langle \mu, \alpha \rangle > c \text{ FOR ALL SIMPLE ROOTS } \alpha, \text{ WE HAVE} \\ \quad C_G^{P,=\mu} : H_{G,I}(W)^{=\mu} \xrightarrow{\sim} H_{M,I}(W)^{=\mu}\end{array}}$

EX $G = SL_2$, $I = \emptyset$, $W = \mathbb{1}$

$$\begin{array}{ccccc} \mathrm{Bun}_G^{\leq n}(\mathbb{F}_q) & \longleftarrow & \mathrm{Bun}_B^{\leq n}(\mathbb{F}_q) & \longrightarrow & \mathrm{Bun}_T^{\leq n}(\mathbb{F}_q) = \coprod_{d \leq n} \underbrace{\mathrm{Bun}_T^{=d}(\mathbb{F}_q)}_{\mathcal{L} \text{ w/ } \deg(\mathcal{L}) = d} \\ \mathcal{E} & \longleftarrow\!\shortmid & (\mathcal{L} \subset \mathcal{E}) & \longmapsto & (\mathcal{L}, \mathcal{E}/\mathcal{L} = \mathcal{L}^{-1}) \end{array}$$

$$\mathrm{Bun}_G^{=n}(\mathbb{F}_q) \xleftarrow[\text{BIJ}]{} \mathrm{Bun}_B^{=n}(\mathbb{F}_q) \xrightarrow[\text{BIJ}]{} \mathrm{Bun}_T^{=n}(\mathbb{F}_q)$$

FOR $n >> 0$

PROP CT MORPHISM COMMUTES w/ HECKE ACTION

$\forall h \in \mathcal{H}_v$

$$\begin{tikzcd} H_{G,I}(W)^{\leq \mu} \arrow[r, "h"] \arrow[d, "C_G^P"'] \arrow[dr, phantom, "\circlearrowleft"] & H_{G,I}(W)^{\leq \mu'} \arrow[d, "C_G^P"] \\ H_{M,I}(W)^{\leq \mu} \arrow[r, "\mathrm{SAT}(h)"] & H_{M,I}(W)^{\leq \mu'} \end{tikzcd}$$

$$\mathrm{SAT}: \mathcal{H}_{G,v} \to \mathcal{H}_{M,v}$$

§4 **THM** $\exists\, \mu_0$ DOM COWT OF G S.T.

$$H_I(W) = \mathcal{H}_v \cdot H_I(W)^{\leq \mu_0}$$

EX $G = SL_2$ $\quad h_G = \mathbb{1}_{G(\mathcal{O}_v)\begin{pmatrix}\pi_v & \\ & \pi_v^{-1}\end{pmatrix} G(\mathcal{O}_v)} \in \mathcal{H}_v$

$$\mathrm{SAT}(h_G) \doteq \mathbb{1}_{T(\mathcal{O}_v)\begin{pmatrix}\pi_v & \\ & \pi_v^{-1}\end{pmatrix} T(\mathcal{O}_v)} + \mathbb{1}_{T(\mathcal{O}_v)\begin{pmatrix}\pi_v^{-1} & \\ & \pi_v\end{pmatrix} T(\mathcal{O}_v)}$$

$$=: h_T + h_T^{-1}$$

$$\begin{tikzcd} H_{G,I}(W)^{\leq n} \arrow[r, "h_G"] \arrow[d, "C_G^B"'] \arrow[dr, phantom, "\circlearrowleft"] & H_{G,I}(W)^{\leq n+1} \arrow[d, "C_G^B"] \\ H_{T,I}(W)^{\leq n} \arrow[r, "h_T + h_T^{-1}"] & H_{T,I}(W)^{\leq n+1} \end{tikzcd}$$

DESCENDS TO

$$\begin{tikzcd} H_G^{=n} = H_G^{\leq n}/H_G^{\leq n-1} \arrow[r, "h_G"] \arrow[d, "C_G^B"'] \arrow[dr, phantom, "\circlearrowright"] & H_G^{\leq n+1}/H_G^{\leq n} = H_G^{=n+1} \arrow[d, "C_G^B"] \\ H_T^{=n} = H_T^{\leq n}/H_T^{\leq n-1} \arrow[r, "h_T + h_T^{-1}"] & H_T^{\leq n+1}/H_T^{\leq n} = H_T^{=n+1} \end{tikzcd}$$

$$H_T^{=d} \xrightarrow[h_T]{\sim} H_T^{=d+1} \Leftarrow \mathrm{Sht}_T^{=d} \xrightarrow[h_T]{\sim} \mathrm{Sht}_T^{=d+1}$$

AND h_T^{-1} ACTS BY 0 AS A MAP

$$H_T^{=n} \longrightarrow H_T^{=n+1}$$

SO FOR $n \gg 0$, C_G^B IS AN ISOM

$\Longrightarrow$ h_G IS AN ISOM ON $H_G^{=n}$

§5

DEF $\quad H_{G,I}(W)^{\mathrm{cusp}} = \bigcap_{P \subsetneq G} \ker(C_G^P)$

PROP 4. $\exists \mu_0$ ST

$$H_{G,I}(W)^{\mathrm{cusp}} \subset \mathrm{Im}\left(H_{G,I}(W)^{\leq \mu_0} \longrightarrow H_{G,I}(W)\right)$$

$\rightarrow$ HAS FINITE DIM.

$$\Rightarrow \quad H_{G,I}(W)^{cusp} \subset H_{G,I}(W)^{Hf}$$

$$H_{G,I}(W)^{cusp} \supset H_{G,I}(W)^{Hf}$$

B/C IF $a \in H_{G,I}(W)^{Hf}$ AND $a \notin H_{G,I}(W)^{cusp}$

$$0 \neq C_G^P(a) \in H_{M,I}(W)^{Hf}$$

CAN USE THIS TO GET A

CONTRADICTION