MSRI

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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Kara Kozia Email/Phone: kkozak@ualberta.ca

Speaker's Name: DAVID NADLER

Talk Title: CATEGORICAL TRACES IN REPRESENTATION THEORY

Date: 4 / 11 / 19 Time: 9 : 30 am / pm (circle one)

Please summarize the lecture in 5 or fewer sentences: THE SPEAKER DISCUSSED TRACE MAPS TO HOCHSCHILD HOMOLOGY, BOTH IN THE CLASSICAL CASE AND THE CATEGORICAL CASE. THE GOAL WAS TO ~~BE~~ EXPLAIN THE "S=T" EQUALITY OF EXCURSION OPERATORS

CHECK LIST

(This is **NOT** optional, we will **not** pay for **incomplete** forms)

- [x] Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- [x] Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - Computer Presentations: Obtain a copy of their presentation
  - Overhead: Obtain a copy or use the originals and scan them
  - Blackboard: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - Handouts: Obtain copies of and scan all handouts
- [x] For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- [x] When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- [x] Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

CATEGORICAL TRACES IN REPRESENTATION THEORY – NADLER

§1 CLASSICAL TRACES

DEFAULT CONTEXT $\quad k$ = FIELD, COMM RING, . . .

DEF $\quad A$ ALG $/k$

1) COCENTER / HOCHSCHILD HOMOLOGY :

$$HH(A) = A/[A,A]$$

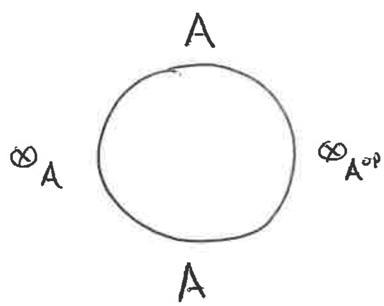
2) TRACE

$$TR : A \longrightarrow HH(A)$$

LEMMA UNIV PROPERTY

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "TR"'] & \arrow[d, phantom, "\circlearrowleft"] & V \\ & HH(A) \arrow[ur, "\exists !"'] & \end{tikzcd} \qquad \forall f \text{ ST } f(ab) = f(ba)$$

REFORMULATION : $\quad HH(A) = A \otimes_{A \otimes_k A^{op}} A$



" ALGEBRA ON A CIRCLE "

THIS IS A CATEGORICAL CONSTRUCTION

ALWAYS BETTER TO DERIVE:

RESOLVE A

$$A \leftarrow \left[A \otimes A \leftleftarrows A \otimes A \otimes A \overset{\leftarrow}{\leftleftarrows} \cdots \right]$$

BAR RES'N AS BIMOD

$$HH(A) \leftarrow \left[A \underset{ba}{\overset{ab}{\leftleftarrows}} A \otimes A \quad \cdots \right]$$

CONVOLUTION

$$(X \times_Y X) * (X \times_Y X)$$

$$\uparrow \Delta_{23}$$

$$X \times_Y X \times_Y X$$

$$\downarrow p_2$$

$$X \times_Y X$$

$$* = p_{2,*} \circ \Delta_{23}^*$$

EX 1) $p : X \twoheadrightarrow Y$ FINITE SETS

$A = \mathcal{O}(X \times_Y X)$ "BLOCK MATRICES OVER Y"

$$\text{TR}: \mathcal{O}(X \times_Y X) \xrightarrow{\Delta_X^*} \mathcal{O}(X) \xrightarrow{p_!} \mathcal{O}(Y)$$

$\mathcal{O}(X \times_Y X) = A$; ONLY "REMEMBER" DIAG'L; $\mathcal{O}(Y) = HH(A)$

2) $H \leq G$ FINITE GPS

$A = (k[H \backslash G / H], *)$ "BLOCK MATRIX ALG"

SET $X = BH \xrightarrow{p} BG = Y$

EXERCISE $H \backslash G / H = BH \times_{BG} BH$

$H \quad G \quad H$

"G-BUNDLE w/ 2 H-REDUCTIONS"

$$\begin{array}{ccccccc}
k[H\backslash G/H] & \xrightarrow{\Delta^*_{BH}} & k[G/_{AD}H] & \xrightarrow{p_!} & k[G/_{AD}G] & = & k[G]^G \\
\parallel & & & & & & \cup \\
A & & & \xrightarrow{TR} & & & HH(A)
\end{array}$$

SPANNED BY IND_H^G

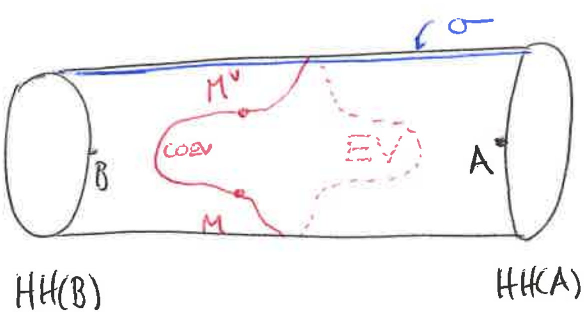
$\Updownarrow$

MORITA INVARIANCE

$\boxed{\text{PROP} \quad HH(A) \text{ IS AN INVARIANT OF } A\text{-MOD}}$ MORE PRECISELY, LET A, B BE ~~ABS~~ ALGS, M DUALIZABLE (A,B)-BIMOD (IE, $B \longrightarrow M^\vee \otimes_A M$, $M \otimes_B M^\vee \longrightarrow A$ SATISFY ZORRO'S AXIOM (sp?))

LEMMA SUCH (A,B)-BIMODS M INDUCE MAPS

$$f_M : HH(B) \longrightarrow HH(A)$$



MORE FORMALLY

$$\begin{array}{ccccccc}
 & & & & \circlearrowleft \otimes_A & & \circlearrowleft \otimes_A \\
B & \xrightarrow{COEV} & M^\vee \otimes_A M & \cong & M \otimes_B M^\vee & \xrightarrow{EV} & A \\
\circlearrowleft \otimes_B & & \circlearrowleft \otimes_B & & & & \\
\parallel & & \parallel & & & & \parallel \\
B \otimes_{B \otimes B^{op}} B & & M^\vee \otimes_{A \otimes B} M & & & & A \otimes_{A \otimes A^{op}} A
\end{array}$$

DEF M DUALIZABLE A-MOD (E.G. DUAL'BLE (A,k)-BIMOD)

$$f_M : k = HH(k) \longrightarrow HH(A)$$

CHARACTER OF M IS $\chi(M) := f_M(1)$

GIVES DIM IN EX 1), CHAR OF A REP IN EX 2)

TWISTED GENERALIZATION

DEF $\sigma = A$-BIMOD

1) σ-COCENTER / σ-HH:

$$HH(A,\sigma) = A \otimes_{A \otimes A^{op}} \sigma$$

2) $TR_\sigma : \sigma \longrightarrow HH(A,\sigma)$ "σ-TRACE"

EXERCISE FORMULATE PREVIOUS RESULTS W/ σ

KEY EX $\sigma : A \longrightarrow A$ AUT $\rightsquigarrow A_\sigma$ ("GRAPH BIMOD")

EX X AFFINE SCHEME, σ AUT OF X. THEN

$$\begin{tikzcd}
A = \mathcal{O}(X) \arrow[r] \arrow[dr, "Res^*"'] & HH(A,\sigma) = \mathcal{O}(X \times_{X \times X} \Gamma_\sigma) \arrow[d, phantom, "\shortparallel"] \\
& \mathcal{O}(X^\sigma)
\end{tikzcd}$$

DEF M DUAL'BLE A-MOD, σ-EQUIV $M \xrightarrow{\phi} M$.

$$\rightsquigarrow \chi_\sigma(M) \in HH(A,\sigma)$$

PUT LINE LABELED BY σ IN DIAG, TWIST ACTION EVERY TIME LINE CROSSES σ

§2 CATEGORICAL TRACES

$\exists$ MANY POSSIBLE SETTINGS...

DEF A MONOIDAL CAT

$$HH(A) = A \otimes_{A \otimes A^{op}} A,$$

IE THE COLIMIT OF THE CAT.

$$HH(A) \leftarrow \left[A \leftleftarrows A \otimes A \Lleftarrow \cdots \right]$$

FROM MONOIDAL STR.

$A \otimes A$ CAN BE CHARACTERIZED BY UNIV PROP

SIMILARLY, IF σ IS AN A-BIMOD, THEN CAN DEFINE

$$HH(A, \sigma)$$

RMK ~~THE DATA OF OBJ OF $HH(A)$ INCLUDES $b \in A$~~
~~w/ FUNCTORIAL ISOM $\alpha_a : ab \xrightarrow{\sim} ba$ $(a \in A)$~~

THE DATA OF MAP $HH(A) \xrightarrow{f} \mathcal{C}$ IS

$$f_0 : A \longrightarrow \mathcal{C} \quad w/$$

$$\alpha_{a,b} : f_0(ab) \xrightarrow{\sim} f(ba) \qquad a, b \in A$$

+ HIGHER CYCLIC IDENTITIES

EX k CHAR 0

1) X REASONABLE STACK, $A = \mathrm{PERF}(X)$, $\sigma : X \longrightarrow X$ AUT

$$A \xrightarrow{TR} HH(A, \sigma) = \mathrm{PERF}(X^{\sigma})$$

$$TR = \mathrm{RES}^*$$

SPECIAL CASE : $\sigma = id$

$$X^\sigma = LX \quad \text{DERIVED LOOP SPACE}$$

1') $X = BG$, $\sigma : G \to G$

$$\begin{array}{ccc} \mathrm{PERF}(BG) & \longrightarrow & \mathrm{PERF}(G/_{\sigma}G) \\ \| & & \\ \mathrm{REP}(G) & \nearrow & \\ & & V \otimes \mathcal{O} \\ V & \longmapsto & \end{array}$$

CAYLEY – HAMILTON : " A σ – EQUIV REP SATISFIES ITS σ – CHAR POLY "

σ – EQUIV REP $\longleftrightarrow$ REP OF $G \rtimes \sigma$

SO

$$\begin{array}{ccc} \mathrm{PERF}(B(G \rtimes \sigma)) & \longrightarrow & \mathrm{PERF}\left(G\rtimes\sigma /_{\mathrm{AD}} G \rtimes \sigma \right) \\ \| & & \\ \mathrm{REP}(G \rtimes \sigma) & \nearrow & \\ & & V \otimes \mathcal{O} \\ V & \longmapsto & \end{array}$$

FIX A PT $(g, \sigma) \in G \rtimes \sigma$, THERE IS A NATURAL ACTION $(g, \sigma) \circlearrowright V \cong V \otimes \mathcal{O}\big|_{(g,\sigma)}$

WOULD LIKE THIS ACTION TO SATISFY CHAR POLY

INTERPRETATION THIS ACTION IS MONODROMY OF CIRCLE ACTION ON $Z(B(G \rtimes \sigma))$ BY ROTATIONS

EX 2) $B \subset G$ BOREL

HECKE CAT $Shv(B \backslash G / B)$ (SAY, CONSTR SHEAVES)

$$
\begin{tikzcd}
A = Shv(B\backslash G/B) \arrow[r, "\Delta^*"] \arrow[rrr, bend right, "TR"'] & Shv(G/_{AD}B) \arrow[r, "P!"] & Shv(G/_{AD}G) & \supset HH(A)
\end{tikzcd}
$$

"HOROCYCLE CORR"

IMAGE IS UNIPOTENT CHAR SHEAVES

FAVORITE MODULE : $A = Shv(B\backslash G/B) \circlearrowright Shv(B\backslash G/G) = M$

$$Shv(B\backslash G/G) = Shv(B\backslash pt)$$

$$\chi(M) = \text{SPRINGER SHEAF}$$

IE $\quad \mu : \mathcal{U}/B \cong \tilde{\mathcal{U}}/G \longrightarrow G/G$

$$\mu_! k_{\mathcal{U}/B} = SPR$$

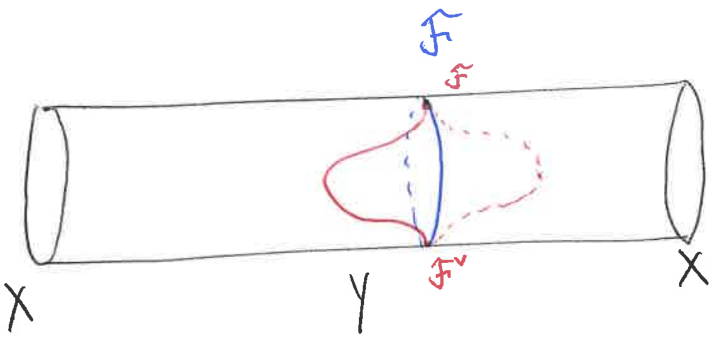
REF BEN-ZVI — NADLER "SECONDARY TRACES"

$$Shv(X \times_Y X) \circlearrowright Shv(X) \quad \rightsquigarrow \quad S_{\mathcal{F}} \in \mathrm{END}(\chi(Shv(X)))$$

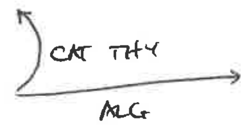
$$\Downarrow$$

$\mathcal{F}$ • DUAL'BLE

$$T_{\mathcal{F}} \in HH(Shv(X \times_Y X))$$



$S_{\mathcal{F}}$ RESULTS FROM



$T_{\mathcal{F}}$ RESULTS FROM