Characters and Sylow 2-subgroups of maximal class

Lecture by Benjamin Sambale
Notes by Dustan Levenstein

Joint with G. Navarro and P. H. Tiep.

1 Introduction

Proposition 1.1. *For a 2-group P the following are equivalent:*

(1) P is (semi)dihedral or quaternion, (but non-abelian, so Klein-4 is excluded)

(2) P has maximal (nilpotency) class,

(3) $|P| \geq 8$ and $|P : P'| = 4$ (Taussky-Todd),

(4) P has exactly 5 rational irreducible characters (Isaacs-Navarro-Sangroniz),

(5) P is non-cyclic and the number of involutions of P is $\equiv 1 \mod 4$ (Alperin-Feit-Thompson),

(6) $P \ncong C_2 \times C_2$ and $\overline{\mathbb{F}}_2 P$ has tame representation type.

This theorem can be partially expanded to an arbitrary group:

Proposition 1.2. *For a finite group G with $P \in \mathrm{Syl}_2(G)$, the following are equivalent:*

(1) P has maximal class,

(2) P is non-cyclic and the number of involutions in G is $\equiv 1 \mod 4$ (Herzog),

(3) $P \ncong C_2 \times C_2$ and the principal 2-block $B_0(G)$ has tame representation type,

(4) missing: character table criterion?

Remark (i) The possible G were classified by Gorenstein-Walter and Alperin-Brauer-Gorenstein.

(ii) The character table does not distinguish the 3 types of P (D_8 vs Q_8).

2 Results

Theorem A. *For $P \in \mathrm{Syl}_2(G)$ the following are equivalent:*

(1) $|P : P'| = 4$,

(2) $|P| = 4$ or there exists $g \in P$ such that $|G : C_G(g)| \equiv 0 \mod 2$ and $\mathbb{Q}(g) := \mathbb{Q}(\chi(g) : g \in \mathrm{Irr}(G)) = \mathbb{Q}(\zeta \pm \zeta^{-1})$ where $\zeta = e^{4\pi i/|P|}$,

(3) $|\mathrm{Irr}_{2'}(B_0(G))| = 4$.

Remark Let $N := N_G(P)$. The Alperin-McKay Conjecture implies

$$\begin{aligned} 4 &= |\operatorname{Irr}_{2'}(B_0(G))| \\ &= |\operatorname{Irr}_{2'}(B_0(N))| \\ &= |\operatorname{Irr}_{2'}(N/O_{2'}(N))| \\ &= |\operatorname{Irr}(N/P'O_{2'}(N))| \end{aligned}$$

holds if and only if $|P : P'| = 4$.

A similar argument applies to $p = 3$. Hence,

Conjecture 2.1. *Let $P \in \operatorname{Syl}_3(G)$. Then $|P : P'| = 9$ iff $|\operatorname{Irr}_{3'}(B_0(G))| \in \{6, 9\}$.*

Easier:

Theorem B. $|\operatorname{Irr}_{3'}(B_0(G))| = 3$ *iff* $|G|_3 = 3$.

Remark (i) Theorem B is well-known for $p = 2$. In fact $|\operatorname{Irr}_{2'}(B_0(G))| \equiv 0 \mod 4$ whenever $|G| \equiv 0 \mod 4$.

(ii) In general $|\operatorname{Irr}_{p'}(B_0(G))|$ does not determine $|G|_p$ or $|P : P'|$:

$$|\operatorname{Irr}(C_{25})| = 25 = |\operatorname{Irr}(C_5^3 \rtimes C_8)|.$$

3 Proofs

Theorem A (2):

$(1) \Rightarrow (2)$: Let $|P| = 2^n \geq 8$ and $|P : P'| = 4$. Then there exists a $g \in P$ of order 2^{n-1} conjugate to g^{-1} or to $g^{-1+2^{n-2}}$ (if $n \geq 4$). Hence $|G : C_G(g)| \equiv 0 \mod 2$ and $|\mathbb{Q}_{2^{n-1}} : \mathbb{Q}(g)| = N_G(\langle g \rangle) : C_G(g)| = 2$.

Since $\mathbb{Q}(g)$ lies in the fixed field of the Galois automorphism $\zeta \mapsto \pm\zeta^{-1}$.

$(2) \Rightarrow (1)$: Let $g \in P$ s.t. $|G : C_G(g)| \equiv 0 \mod 2$ and $\mathbb{Q}(\zeta \pm \zeta^{-1}) = \mathbb{Q}(g) \subseteq \mathbb{Q}_{|\langle g \rangle|}$. Then P is a non-abelian group and we may assume that $n > 3$ (the case $n = 3$ being easy). Then $\operatorname{Gal}(\mathbb{Q}(g) \mid \mathbb{Q})$ is cyclic and therefore $\mathbb{Q}(g)$ is not a cyclotomic field. Therefore $\mathbb{Q}(g) \subsetneq \mathbb{Q}_{|\langle g \rangle|}$. Hence $|\langle g \rangle| = 2^{n-1}$. If P does not have maximal class, then g is conjugate to $g^{1+2^{n-2}}$. This yields the contradiction.

$$\mathbb{Q}(g) \subseteq \mathbb{Q}(\zeta^2).$$

$(1) \Rightarrow (3)$: well-known (Brauer, Olsson)

$(3) \Rightarrow (1)$: uses the classification of finite simple groups (CFSG).

Easy cases: If $B_0(G)$ is the only block of maximal defect, then the claim follows from Malle-Späth (McKay Conjecture for $p = 2$.)

Remark The proof of Theorem B (2) also relies on CFSG.

4 Open problems

Let B be any p-block with defect group D and $k(B) := |\operatorname{Irr}(B)|$, $k_0(B) := |\{\chi \in \operatorname{Irr}(B) : \chi(1)_p = |G : D|_p\}|$.

Question 1: If $p = 2$ and $k_0(B) = 4$, then is it true that$|D : D'| = 4$?
Question 2: If $p = 3$ and $k_0(B) = 3$, then $|D| = 3$?
Question 3: If $p = 2$ and $k(B) =$ then $|D| = 4$?
Question 4: If $k(B) = 3$ then $p = |D| = 3$?