Intersections of Finite Sets: Geometry and Topology

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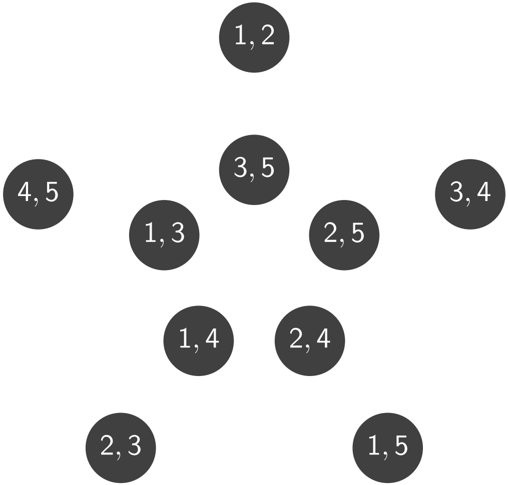
Cornell University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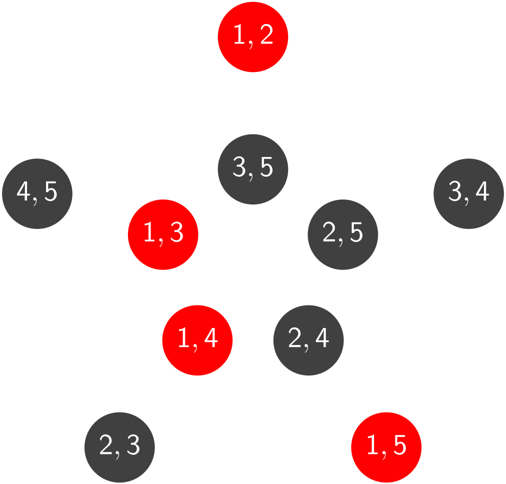
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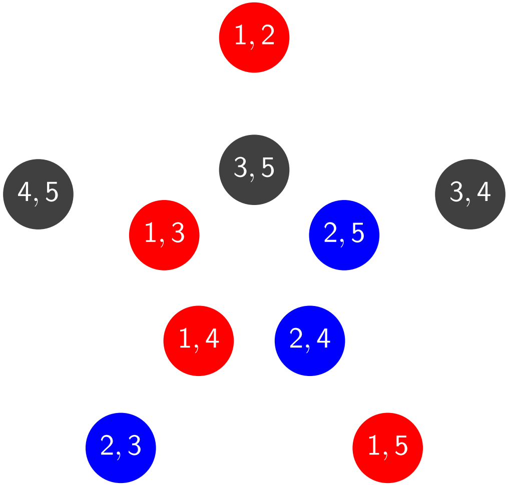
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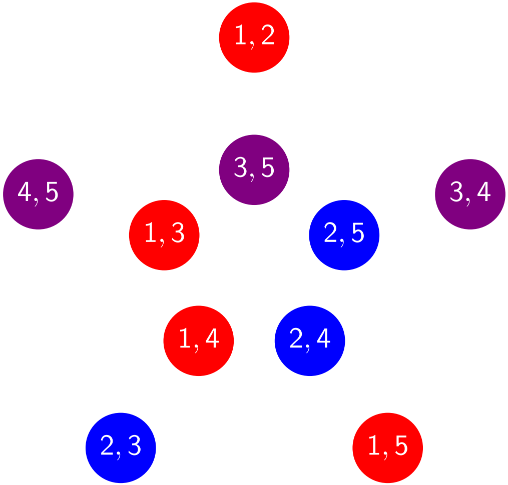
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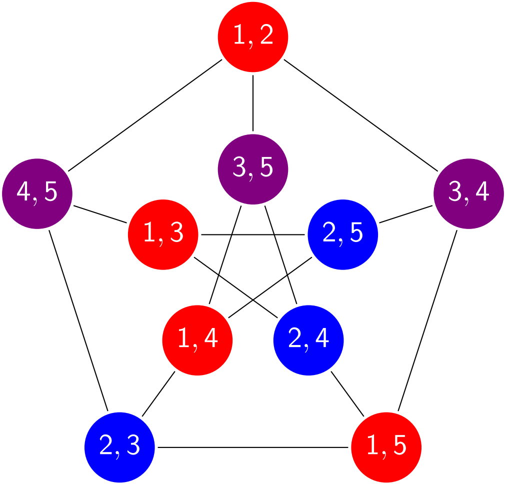

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Kneser's conjecture – Lovász 1978

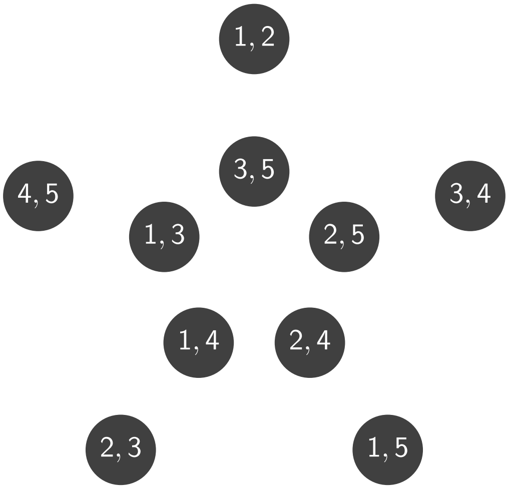
k-subsets of $[n] = \{1, \ldots, n\}$

Then $\chi(\mathrm{KG}(k, n)) = n - 2(k - 1)$

Immediate Questions

- Fewer sets? $\mathcal{F} \subsetneq \binom{[n]}{k}$ with $\chi(\mathrm{KG}(\mathcal{F})) = n - 2(k-1)$
- Generalize to more than two disjoint sets?
  Graphs $\longrightarrow$ hypergraphs
- Bounds for arbitrary set systems $\mathcal{F}$?

Fewer sets?



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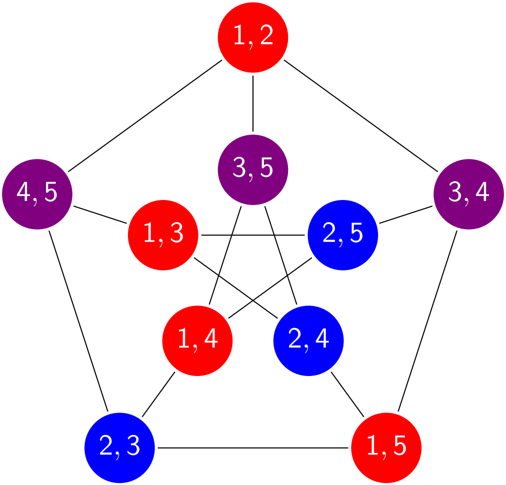
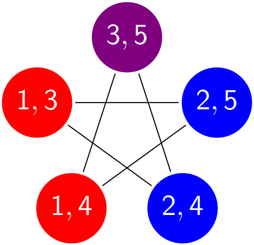


Fewer sets?



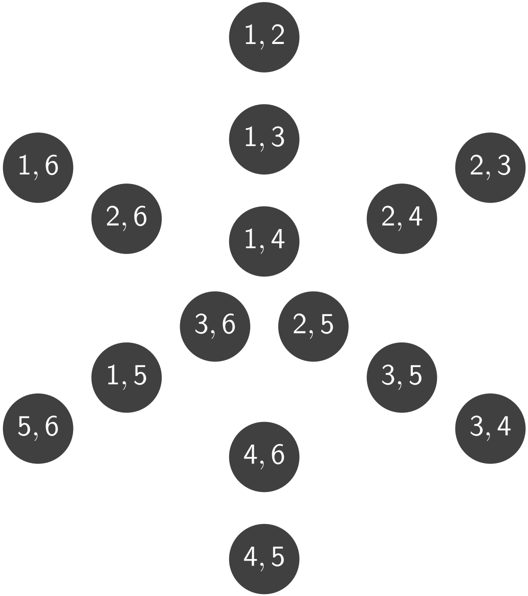
Schrijver's theorem – 1978

$\mathcal{F} \subset \binom{[n]}{k}$ the stable sets,
i.e., no two adjacent elements in cyclic order.

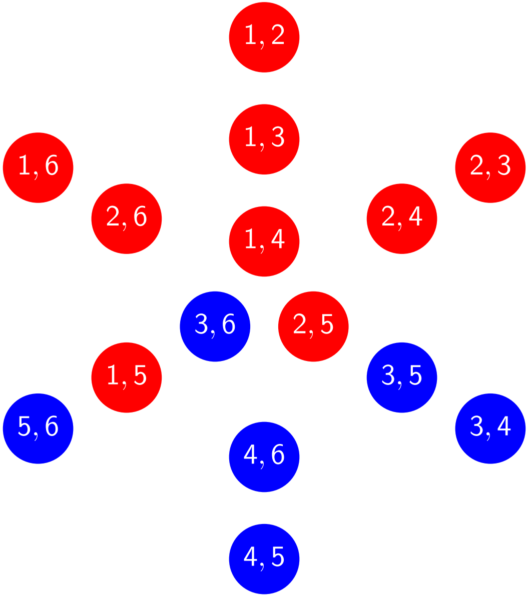
Then $\chi(\mathrm{KG}(\mathcal{F})) = n - 2(k-1)$

This is optimal

Generalize to more than two disjoint sets?



Generalize to more than two disjoint sets?



Erdős conjecture

How many colors are needed to color $\binom{[n]}{k}$ such that any r pairwise disjoint sets have at least two colors?

Denote the minimal number of colors by $\chi(\mathrm{KG}^r(k, n))$

Alon, Frankl, Lovász – 1986

$$\chi(\mathrm{KG}^r(k,n)) = \left\lceil \frac{n - r(k-1)}{r-1} \right\rceil$$

Examples

$\binom{[6]}{2}$ by 2 colors $\longrightarrow$ no 3 monochromatic pairwise disjoint sets

$\binom{[8]}{2}$ by 3 colors $\longrightarrow$ no 3 monochromatic pairwise disjoint sets

$\binom{[10]}{2}$ by 4 colors $\longrightarrow$ no 3 monochromatic pairwise disjoint sets

Bounds for arbitrary set systems $\mathcal{F}$?

Given $\mathcal{F} \subset 2^{[n]}$

Find lower bounds for $\chi(\mathrm{KG}^r(\mathcal{F}))$

Dol'nikov – 1988

Given $\mathcal{F} \subset 2^{[n]}$

Then $\chi(\mathrm{KG}(\mathcal{F})) \geq \mathrm{cd}(\mathcal{F})$

$$\mathrm{cd}(\mathcal{F}) = n - \max\{|A \cup B| \ : \ F \not\subset A \text{ and } F \not\subset B \text{ for all } F \in \mathcal{F}\}$$

Colorability defect

$$\chi(\mathrm{KG}(\mathcal{F})) \geq \mathrm{cd}(\mathcal{F})$$

This generalizes Kneser's conjecture:

$$\mathrm{cd}(\binom{[n]}{k}) = n - 2(k-1)$$

However, if $\mathcal{F} \subset \binom{[n]}{k}$ is the collection of stable sets

$$\mathrm{cd}(\mathcal{F}) = n - 4(k-1)$$

Kříž – 1992

Given $\mathcal{F} \subset 2^{[n]}$

Then $\chi(\mathrm{KG}^r(\mathcal{F})) \geq \left\lceil \frac{\mathrm{cd}^r(\mathcal{F})}{r-1} \right\rceil$

$\mathrm{cd}^r(\mathcal{F}) = n - \max\{|\bigcup_{i=1}^r A_i| \ : \ F \not\subset A_i \text{ for all } F \in \mathcal{F} \text{ and } i \in [r]\}$

Kneser's conjecture and its generalizations

- Lovász: $\chi(\mathrm{KG}(k,n)) = n - 2(k-1)$
- Schrijver: $\chi(\mathrm{KG}(k,n)_{\mathrm{stab}}) = n - 2(k-1)$
- Alon, Frankl, Lovász: $\chi(\mathrm{KG}^r(k,n)) = \left\lceil \frac{n-r(k-1)}{r-1} \right\rceil$
- Dol'nikov, Kříž: $\chi(\mathrm{KG}^r(\mathcal{F})) \geq \left\lceil \frac{\mathrm{cd}^r(\mathcal{F})}{r-1} \right\rceil$

All imply Kneser's conjecture

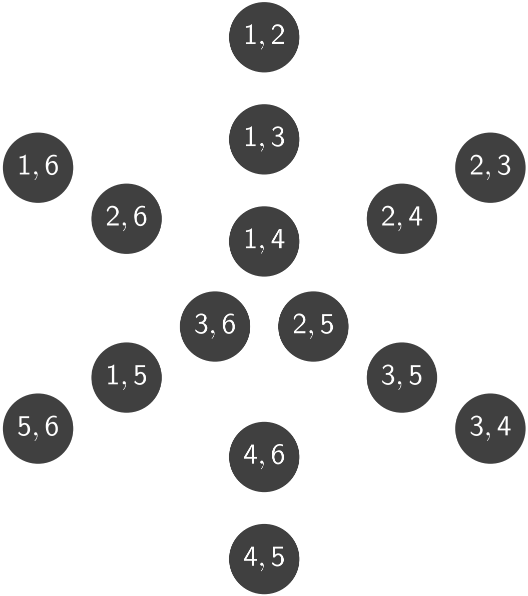
Dol'nikov–Kříž implies Alon–Frankl–Lovász, but not Schrijver

Questions

- What is the general principle here?
- Is there a stable set version for $\chi(\mathrm{KG}^r(k, n))$?
- Is there a stable set version for $\chi(\mathrm{KG}^r(\mathcal{F}))$?
- Is there a common generalization of all of these results?

We have to take the topology of the set systems into account

Stable set version of Alon–Frankl–Lovász?



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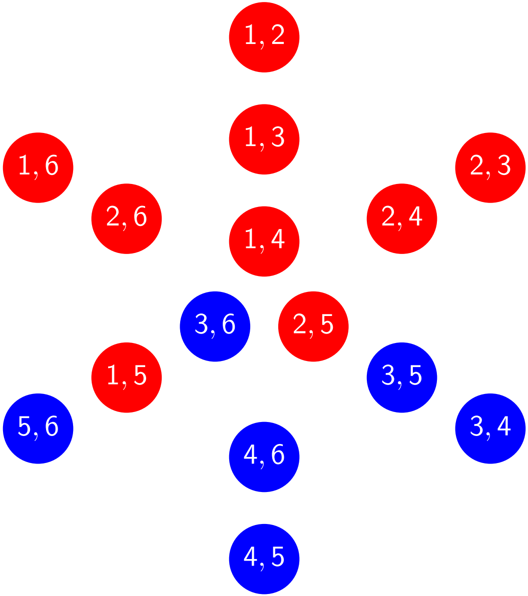

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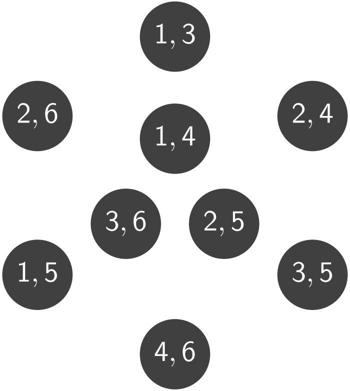

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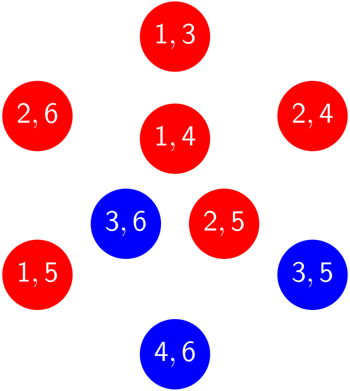


Stable set version of Alon–Frankl–Lovász?

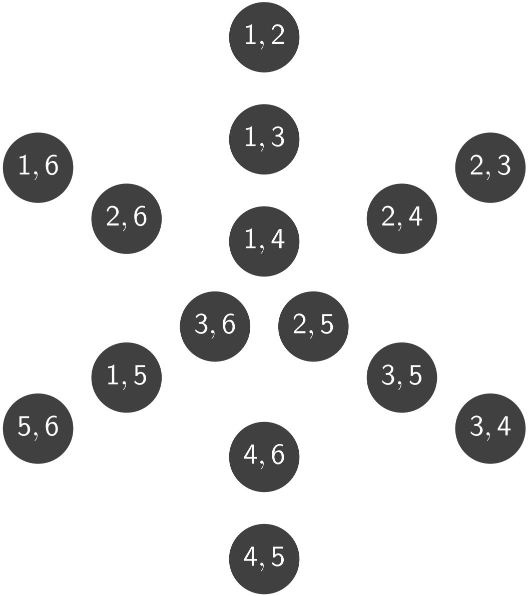
Actually, a conjecture of Ziegler states
that one can restrict to r-stable sets

Meunier – 2011

$\mathcal{F} \subset \binom{[n]}{k}$ the collection of almost stable k-sets,
i.e., $F \in \mathcal{F}$ never contains i and $i+1$

Then $\chi(\mathrm{KG}^r(\mathcal{F})) = \left\lceil \frac{n-r(k-1)}{r-1} \right\rceil$

Stable set version of Alon–Frankl–Lovász?



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Meunier's theorem is off by $\binom{n-2}{k-2}$ additional sets

Alishahi and Hajiabolhassan – 2015

Let r be even or $n \not\equiv k \mod r-1$

Then $\chi(\mathrm{KG}^r(k,n)_{\mathrm{stab}}) = \left\lceil \frac{n-r(k-1)}{r-1} \right\rceil$

F. – 2017

$$\chi(\mathrm{KG}^r(k,n)_{\mathrm{stab}}) = \left\lceil \frac{n-r(k-1)}{r-1} \right\rceil$$

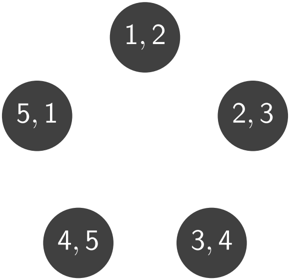
Stable set theorem for arbitrary set systems?

$\mathcal{F} \subset 2^{[n]}$ an arbitrary set system

Is $\chi(\mathrm{KG}^r(\mathcal{F}_{\mathrm{stab}})) \geq \left\lceil \frac{\mathrm{cd}^r(\mathcal{F})}{r-1} \right\rceil$?

Stable set theorem for arbitrary set systems?

No.



$\mathcal{F}_{\mathrm{stab}} = \emptyset$, but $\mathrm{cd}(\mathcal{F}) = 1$

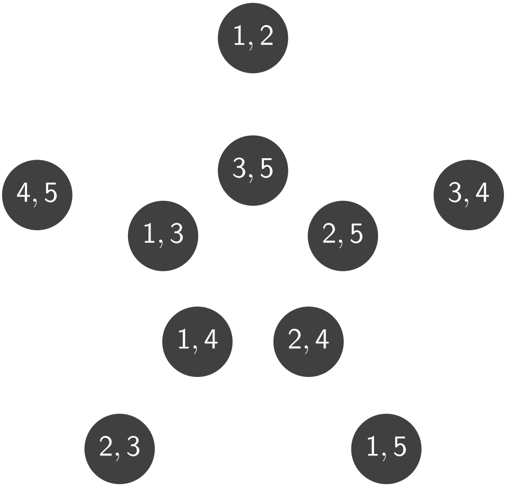
F. – 2017

For $r \geq 3$

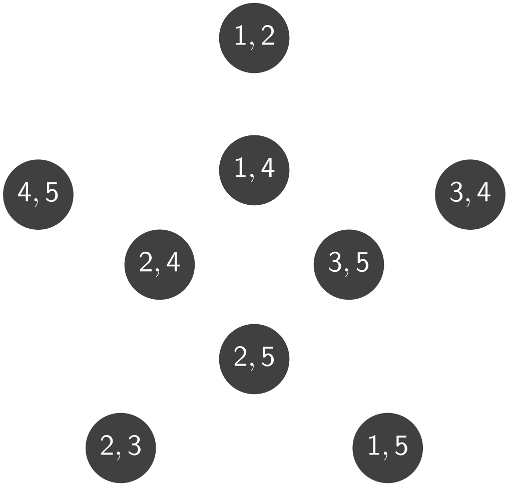
$$\chi(\mathrm{KG}^r(\mathcal{F}_{\mathrm{stab}})) \geq \left\lceil \frac{\mathrm{cd}^r(\mathcal{F})}{r-1} \right\rceil$$

and if $\mathcal{F}' \subset \mathcal{F}$ consists of all almost stable sets

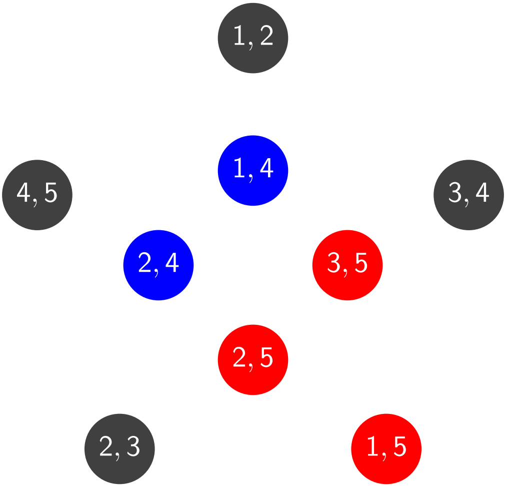
then $\chi(\mathrm{KG}(\mathcal{F}')) \geq \mathrm{cd}(\mathcal{F})$



$\mathrm{cd} = 3$



$$\mathrm{cd} = 2$$



$\mathrm{cd} = 2$

Dictionary

We will set up a dictionary

Intersection patterns of finite sets

$\longleftrightarrow$

Intersection patterns of convex sets

Elementary results on one side translate into results on the other side that were previously believed to be difficult

Dictionary

We will set up a dictionary

Intersection patterns of finite sets

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Intersection patterns of convex sets

Elementary results on one side translate into results on the other side that were previously believed to be difficult

A proof of Kneser's conjecture

Suppose $\chi(\mathrm{KG}(2,5)) \leq 2$

Embed [5] into $\mathbb{R}$

Extend to a continuous map $\Delta_4 \longrightarrow \mathbb{R}^3$ by mapping barycenters of any face containing a red edge to e_2 and barycenters of faces with only blue edges to e_3

No two disjoint faces intersect in the image—contradiction

Topological Radon theorem – Bárány, Bajmoczy 1978

Any continuous map $f\colon \Delta_{d+1} \longrightarrow \mathbb{R}^d$
identifies points from two disjoint faces

A proof of Kneser's conjecture

Suppose $\chi(\mathrm{KG}(k, n)) \leq n - 2(k-1) - 1$

Embed $\Delta_{n-1}^{(k-2)}$ into $\mathbb{R}^{2k-3}$

Extend to a continuous map $\Delta_{n-1} \longrightarrow \mathbb{R}^{2k-3} \oplus \mathbb{R}^{n-2(k-1)-1}$

$$2k - 3 + n - 2(k-1) - 1 = n - 2$$

No two disjoint faces intersect in the image—contradiction

Extending this approach

Given set system $\mathcal{F} \subset 2^{[n]}$
find simplicial complex K with missing faces (minimal nonfaces) $\mathcal{F}$

Embed K into the smallest possible $\mathbb{R}^d$

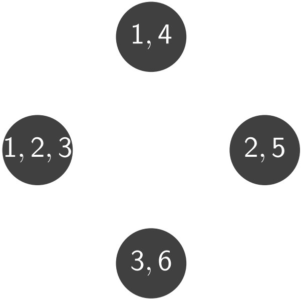
Extend to continuous map $\Delta_{n-1} \longrightarrow \mathbb{R}^{n-2}$
where no two disjoint faces overlap

Thus $\chi(\mathrm{KG}(\mathcal{F})) \geq n - 1 - d$

Extending this approach

Earlier such results are due to Sarkaria
(with significantly more involved proofs)

Example



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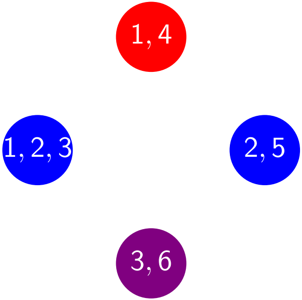
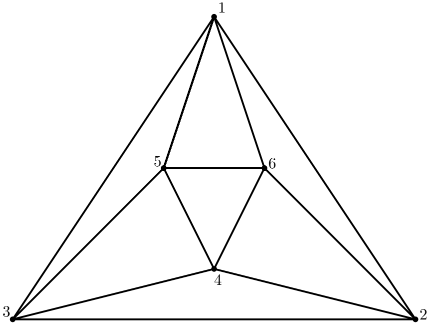


Example

$$\mathrm{cd}(\mathcal{F}) = 0$$



$$\chi(\mathcal{F}) \geq 6 - 1 - 2 = 3$$

r disjoint sets — avoiding r-fold intersections

Given $\mathcal{F} \subset 2^{[n]}$

Let K be the simplicial complex with missing faces $\mathcal{F}$

Find continuous $K \longrightarrow \mathbb{R}^d$ such that r pairwise disjoint faces of K never have overlapping images.

Tverberg-type theory

Bárány, Shlosman, Szűcs; Özaydin – 1981/87
$f : \Delta_{(r-1)(d+1)} \longrightarrow \mathbb{R}^d$ continuous, r a power of a prime
then there are $\sigma_1, \ldots, \sigma_r$ pairwise disjoint with $\bigcap_i f(\sigma_i) \neq \emptyset$.

Tverberg-type results for specific families of complexes due to
Van Kampen; Flores; Sarkaria; Volovikov; Vrećica and Živaljević;
Hell; Engström; Blagojević, Matschke, and Ziegler, ...

Tverberg-type theory — a general principle

Blagojević, F., Ziegler – 2014
There are general combinatorial conditions on
a simplicial complex K
such that any continuous map $K \longrightarrow \mathbb{R}^d$ must identify
points from r pairwise disjoint faces of K,
where r is a power of a prime.

Intersection patterns of stable sets

Color $\binom{[n]}{k}_{\mathrm{stab}}$ such that any r pairwise disjoint sets receive at least two colors

$$\chi(\mathrm{KG}^r(\binom{[n]}{k}_{\mathrm{stab}})) = \left\lceil \frac{n-r(k-1)}{r-1} \right\rceil$$

Intersection patterns of stable sets

The simplicial complex K with missing faces $\binom{[n]}{k}_{\mathrm{stab}}$ is the boundary of a cyclic $(2k-2)$-polytope on n vertices

Schrijver's theorem follows from: cone over K embeds into $\mathbb{R}^{2k-2}$

Thus $\chi(\mathrm{KG}(\binom{[n]}{k}_{\mathrm{stab}})) \geq n - 2(k-1)$

Intersection patterns of stable sets

Map $\partial C_{2k-2}(n) * \Delta_t$ to $\mathbb{R}^d$ without r-fold points of intersection among pairwise disjoint faces

Vertices sufficiently far apart on stretched moment curve (Bukh, Loh, Nivasch)

Use a proper extension of the topological Tverberg theorem due to Hell

Induct on prime divisors of r

Topological colorability defect

$\mathrm{tcd}^r(\mathcal{F})$ is the maximum over $N - (r-1)(d+1)$

K has N vertices, missing faces $\mathcal{F}$, and
there is a continuous map $K \longrightarrow \mathbb{R}^d$ without r-fold point of coincidence among pairwise disjoint faces

Topological lower bounds; F. – 2017

$\mathrm{tcd}^r(\mathcal{F}) \geq \mathrm{cd}^r(\mathcal{F})$ by a general position map

For r a power of a prime

$$\chi(\mathrm{KG}^r(\mathcal{F})) \geq \left\lceil \frac{\mathrm{tcd}^r(\mathcal{F})}{r-1} \right\rceil$$

It follows from work of Mabillard and Wagner that this fails for every r that is not a prime power

Transversal extension; F. – 2017

Let r be a prime. Partition $[n]$ such that each part has size at most $r-1$

Then $\chi(\mathrm{KG}^r(\mathcal{F}_{\mathrm{transversal}})) \geq \left\lceil \frac{\mathrm{tcd}^r(\mathcal{F})}{r-1} \right\rceil$

Key idea: use optimal colored Tverberg theorem of Blagojević, Matschke, and Ziegler

This is a common generalization of the results of Schrijver, Alon–Frankl–Lovász, Dol'nikov, and Kříž

s-stable sets

$\sigma \subset [n]$ is *s-stable* if any two elements in σ are at distance at least s in cyclic order

Conjecture (Ziegler):
$$\chi(\mathrm{KG}^r(k,n)_{r-\mathrm{stab}}) = \left\lceil \frac{n-r(k-1)}{r-1} \right\rceil$$

Known if $r = 2$ (Schrijver)
and thus if $r = 2^t$ (Alon–Drewnowski–Łuczak)

Apart from this reduction, nothing was known for s-stable sets, $s > 2$

s-stable sets

F. – 2017:

$r > 6s - 6$ a prime power, then

$$\chi(\mathrm{KG}^r(k,n)_{s-\mathrm{stab}}) = \left\lceil \frac{n-r(k-1)}{r-1} \right\rceil$$

Relies on a topological connectivity result due to Engström

In fact, for the same parameters and $\mathcal{F} \subset 2^{[n]}$

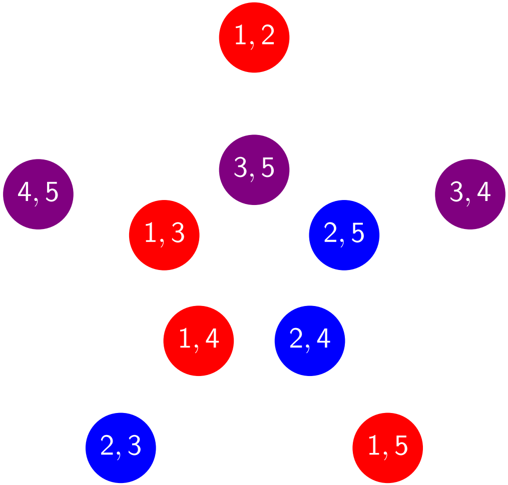
$$\chi(\mathrm{KG}^r(\mathcal{F}_{s-\mathrm{stab}})) \geq \left\lceil \frac{\mathrm{tcd}^r(\mathcal{F})}{r-1} \right\rceil$$

Generalizing pairwise intersections

Kneser conjecture seeks to split $\binom{[n]}{k}$ into color classes such that sets of the same color intersect pairwise.

What if we want any collection of r not necessarily distinct sets that is s-wise disjoint to receive at least 2 colors? $\longrightarrow \mathrm{KG}^r_{s-1}(k,n)$

Generalized Kneser conjecture (Sarkaria):
$\chi(\mathrm{KG}^r_{s-1}(k,n)) = \left\lceil \frac{(s-1)n-r(k-1)}{r-1} \right\rceil$



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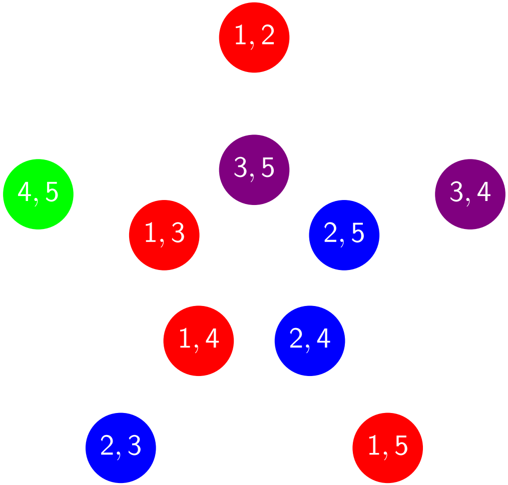

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Generalizing pairwise intersections

Generalized Kneser conjecture (Sarkaria):

$$\chi(\mathrm{KG}^r_{s-1}(k,n)) = \left\lceil \frac{(s-1)n - r(k-1)}{r-1} \right\rceil$$

True if r is prime (Sarkaria – 1990)

True for any r if $s \leq 5$ and for $r = 2^t$
(Aslam, Chen, Coldren, F., Setiabrata – 2017)

Generalizing pairwise intersections

Partition $[(s-1)n]$ into $P_1, \ldots, P_n$ such that
$\ell \in P_i \iff \ell \equiv i \mod n$

Then $\chi(\mathrm{KG}^r_{s-1}(k,n)) \geq \chi(\mathrm{KG}^r(\binom{[(s-1)n]}{k}_{\mathrm{transversal}}))$

This is at least $\left\lceil \frac{(s-1)n - r(k-1)}{r-1} \right\rceil$ for r a prime.
There is an induction on prime divisors that gives the result for $s \leq 5$.

Generalizing pairwise intersections

Aslam, Chen, Coldren, F., Setiabrata – 2017:
Let $r = 2^{\alpha_0} p_1^{\alpha_1} \cdots p_d^{\alpha_d}$ with the odd primes $p_1, \ldots, p_d$ distinct, $\alpha_i \geq 0$. Let $b := 2^{\alpha_0}(p_1 - 1)^{\alpha_1} \cdots (p_d - 1)^{\alpha_d}$.

Then for all $2 \leq s \leq \min\{r, b+1\}$,

$$\chi(KG^r_{s-1}(n,k)) = \left\lceil \frac{(s-1)n - r(k-1)}{r-1} \right\rceil.$$

Geometry of vertex-critical subgraphs

$\chi(\mathrm{KG}(k,n)_{\mathrm{stab}}) = n - 2(k-1)$ follows from the existence of cyclic polytopes

More generally, every even-dimensional neighborly polytope gives a small subgraph of $\mathrm{KG}(k,n)$ of the same chromatic number