SIEVE WEIGHTS AND THEIR SMOOTHINGS

Notation: $n \sim x$ if $x < n \leq 2x$
Counting twin primes

$$\#\{n \sim x : n, n+2 \text{ prime }\} = \#\{n \sim x : (a,m) = 1\},$$

where

$$m = \prod_{p \leq \sqrt{2x}} p, \quad a = n(n+2).$$

Mobius inversion:

$$1_{(a,m)=1} = \sum_{d|(a,m)} \mu(d)$$

gives a formula

$$\begin{aligned} &\#\{n \sim x : n, n+2 \text{ prime }\} \\ =&\sum_{n \sim x} \sum_{d|(a,m)} \mu(d), \\ =&\sum_{d|m} \mu(d) \sum_{\substack{n \sim x \\ n(n+2) \equiv 0 \pmod d}} 1 \end{aligned}$$

$m = \prod_{p \leq \sqrt{2x}} p \approx e^{\sqrt{2x}}$, problems arise when d is big.
want to find a strong correlation:

$$1_{(a,m)=1} \approx \sum_{d|(a,m), d \leq D} \rho_d, \ \rho_d = 0, \text{ if } d > D = x^\theta$$

Combinatorial sieve:

$$\rho_d = \mu(d) 1_{d \in \mathcal{D}}, \ d = p_1 \dots p_r$$

Selberg's sieve :

$$1_{(a,m)=1} \leq \left(\sum_{d|(a,m)} \lambda_d \right)^2, \ \lambda_1 = 1, \lambda_d = 0, \text{ if } d > R, D = R^2$$

$$\begin{aligned}\#\{n \sim x : n, n+2 \text{ prime } \} &\leq \sum_{n \sim x} \left(\sum_{d|(a,m)} \lambda_d \right)^2 \\ &= \sum_{\substack{d_1, d_2 \\ d=[d_1,d_2]}} \lambda_{d_1} \lambda_{d_2} \sum_{\substack{n \sim x \\ n(n+2) \equiv 0 \pmod{d}}} 1 \\ &= \sum_{d_1, d_2 \leq R} \lambda_{d_1} \lambda_{d_2} \left(s \frac{\rho(d)}{d} + O(\rho(d)) \right),\end{aligned}$$

where $\rho(d) = \#\{n \in \mathbb{Z}/d\mathbb{Z} : n(n+2) \equiv 0 \pmod{d}\}$.

$$\lambda_d \approx c\mu(d) \left(\frac{\log R/d}{\log R} \right)^2 1_{d \leq R}$$

$$\#\{n \sim x : n, n+2 \text{ prime } \} \leq (8 + o(1)) \frac{c_2 x}{\log^2 x},$$

where $c_2 = 2 \prod_{p \geq 3} (1 - \frac{2}{p})(1 - \frac{1}{p})^{-2}$, twin prime constant

Counting k-tuple primes

$$\#\{n \sim x : n + h_1, n + h_2, \ldots, n + h_k \text{ all prime } \}$$

need

$$\lambda_d \approx c\mu(d) \left(\frac{\log(R/d)}{R} \right)^k 1_{d \leq R}$$

General

$$\mathcal{M}_f(n; R) = \sum_{d|n} \mu(d) f\left(\frac{\log d}{\log R} \right)$$

$supp(f) \subset (-\infty, 1], f(1) = 1 \Longrightarrow \mathcal{M}_f(n; R) = 1$ if $p \mid n \implies p > R$

$$\#\{n \leq x : p \mid n \implies p > R\} \approx \frac{x}{\log R}$$

$$\mathcal{M}(n; R) = \sum_{d|n} \mu(d)$$

If $n = 2m, 2 \nmid m$,

$$\begin{aligned}\mathcal{M}(n;R) &= \sum_{d|n} \mu(d) \\ &= \sum_{\substack{d|m \\ d\le R}} \mu(d) + \sum_{\substack{d|m \\ d\le R/2}} \mu(2d) \\ &= \sum_{\substack{d|m \\ R/2<d\le R}} \mu(d) \\ &= \pm 1 \text{ if } m \text{ has a unique square free divisor in (R/2,R]}\end{aligned}$$

Ford:

$$\#\{n \le x : n \text{ has a unique square free divisor in (R/2,R]}\} \asymp \frac{x}{(\log R)^{\delta}(\log\log R)^{3/2}},$$

where $\delta = 1 - \frac{1+\log\log 2}{\log 2} = 0.086071\cdots < 1$. This implies that

$$\sum_{n\le x} \mathcal{M}(n;R)^2 \gg \frac{x}{(\log R)^{\delta+o(1)}},$$

while

$$\#\{n \le x : p \mid n \implies p > R\} \approx \frac{x}{\log R}$$

If $f \in C^1(\mathbb{R})$, $n = p^v m,\ p \nmid m$, then

$$\begin{aligned}\mathcal{M}_f(n;R) &= \sum_{d|m} \mu(d) f\left(\frac{\log d}{\log R}\right) - \sum_{d|m} \mu(d) f\left(\frac{\log(dp)}{\log R}\right) \\ &= -\int_0^{\frac{\log p}{\log R}} \sum_{d|m} \mu(d) f'\left(u + \frac{\log d}{\log R}\right) du\end{aligned}$$

If $f \in C^A(\mathbb{R})$ and $n = p_1^{v_1}\cdots p_A^{v_A} m$, ($p_j$ is the j-th smallest prime factor of n),

$$\begin{aligned}\mathcal{M}_f(n;R) &= (-1)^A \int_0^{\frac{\log p_1}{\log R}} \cdots \int_0^{\frac{\log p_A}{\log R}} \sum_{d|m} \mu(d) f\left(u_1 + \cdots + u_A + \frac{\log d}{\log R}\right) d\underline{u} \\ &\lesssim \frac{\log p_1}{\log R} \cdots \frac{\log p_A}{\log R} \mathcal{M}_{f^{(A)}}(m;R)\end{aligned}$$

If n is typical, then

$$\mathcal{M}_f(n;R) \lesssim \frac{\mathcal{M}_{f^{(A)}}(m;R)}{(\log R)^A}$$

(soft) Conjecture:

$$\sum_{n\le x}\mathcal{M}_f(n;R)^{2k}\lesssim \max\left\{\underset{\substack{\uparrow\\ \text{almost primes}}}{\frac{x}{\log R}},\ \underset{\substack{\uparrow\\ \text{typical integers}}}{\frac{\sum_{m\le x}\mathcal{M}(n;R)^{2k}}{(\log R)^{2k}}}\right\}.$$

Conjecture :

$$\sum_{n\le x}\mathcal{M}(n;R)\sim c_k x(\log R)^{E_k}$$

need $E_k-2kA<0\Leftrightarrow A>\frac{E_k}{2k}$ for $\mathcal{M}_f(n;R)^{2k}$ to behave like a sieve weight.

Two questions:

1. $E_k=?$
2. Is (soft) conjecture true?

For the second question, we have

Theorem 1 (GKM '16)**.** *Suppose $f\in C^A(\mathbb{R})$, $f,f',\dots,f^{(A)}$ are uniformly bounded, supp$(f)\subset(-\infty,1]$,*

a) *If $A>\frac{E_k}{2k}+1$, then*

$$\sum_{\substack{n\le x\\ \exists p|n,p\le R^\alpha}}\mathcal{M}_f(n;,R)^{2k}\ll\alpha^{3/2}\frac{x}{\log R},\ \left(x\ge R\ge\alpha,\ 1\ge\alpha\ge\frac{2}{\log R}\right)$$

$$\sum_{n\le x}\mathcal{M}_f(n;R)^{2k}\sim c_{k,f}\frac{x}{\log R},\qquad\left(x\ge R^{2k}\log^2 R,R\to\infty\right)$$

b) *If $A\le\frac{E_k}{2k}+1$, then*

$$\sum_{n\le x}\mathcal{M}_f(n;R)^{2k}\ll x(\log R)^{E_k-2k(A-1)}$$

For the first question on E_k:

(1) $k=1$, $E_1=0$, (Dress, Iwaniec and Tenenbaum, '83)
(2) $k=2$, $E_2=0$, (Motohashi, '04)
(3) $\forall k$, E_k exists, (Balazard, Naimi, and Pétermann, '08, de la Bréteche, '01)

$$\begin{aligned}\sum_{n\le x}\mathcal{M}(n;R)^{2k} &= \sum_{n\le x}\Bigg(\sum_{d|n,d\le R}\mu(d)\Bigg)^{2k}\\ &= \sum_{d_1,\dots,d_{2k}\le R}\mu(d_1)\cdots\mu(d_{2k})\sum_{\substack{n\le x\\ d_1\cdots d_{2k}|n}}1\\ &= x\sum_{d_1,\dots,d_k\le R}\frac{\mu(d_1)\cdots\mu(d_k)}{[d_1,\dots,d_{2k}]}+O(R^{2k})\end{aligned}$$

$$x\ge R^{2k+\epsilon}\sim\frac{x}{(2\pi i)^{2k}}\int_{\Re(s_1)=\frac{1}{\log R}}\cdots\int_{\Re(s_{2k})=\frac{1}{\log R}}\sum_{d_1,\dots,d_{2k}}\frac{\mu(d_1)\cdots\mu(d_{2k})}{[d_1,\dots,d_{2k}]}\prod_{i=1}^{2k}\frac{R^{s_i}}{s_i d_i^{s_i}}d\underline{s}$$

$$\begin{aligned}[d_1,\dots,d_{2k}]=p \implies &\exists I\neq\emptyset, s.t.\\ &d_i=p, i\in I\\ &d_j=1, j\notin I\end{aligned}$$

$$\rightsquigarrow(-1)^{|I|}\frac{1}{p^{1+s_I}},\ \ s_I=\sum_{i\in I}s_i$$

$$\sum_{d_1,\dots,d_{2k}}\frac{\mu(d_1)\cdots\mu(d_{2k})}{[d_1,\dots,d_{2k}]}\prod_{i=1}^{2k}\frac{R^{s_i}}{s_i d_i^{s_i}}=F(\underline{s})\frac{\prod_{I\ \text{even}\neq\emptyset}\zeta(1+s_I)}{\prod_{I\ \text{odd}}\zeta(1+s_I)}$$

Shift s_{2k} to the left, poles when $s_I=0$, $2k\in I$ I even,

$$\text{order } = \sum_{I\neq\emptyset,2k\in I,s_I=0}(-1)^{|I|}$$

Denote $s_{2k}=\mathcal{L}(s_1,\dots,s_{2k})$, where $\mathcal{L}$ is a linear form. Shift s_{2k-1}, pick poles when $s_I=0$ and $2k-1\in I$. $s_{2k-1}=\mathcal{L}_1(s_1,\dots,s_{2k-2})$ $\rightsquigarrow s_{2k}=\tilde{\mathcal{L}}(s_1,\dots,s_{2k-2})$. Shift $2k-m$ variables, $s_i=\sum_{j=1}^m a_{i,j}s_j$,

$$s_I=0\Leftrightarrow\sum_{j=1}^m(\sum_{i\in I}a_{i,j})s_j=0\Leftrightarrow\sum_{i\in I}a_{i,j}=0,\forall j$$

$$E_k=\max_{\substack{m\ge1\\ A\text{ contains } I_m}}\left(\sum_{\substack{I\neq\emptyset,\subset\{1,\dots,2k\}\\ \sum_{i\in I}a_{i,j}=0\forall j}}(-1)^{|I|}-2k+m\right)$$

optimal: $m=1,(a_1,\dots,a_{2k})=(1,-1,\dots,1,-1)$

$$E_k=\binom{2k}{k}-2k$$