Cluster Algebras and Exact Lagrangian Surfaces

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Microlocal sheaves

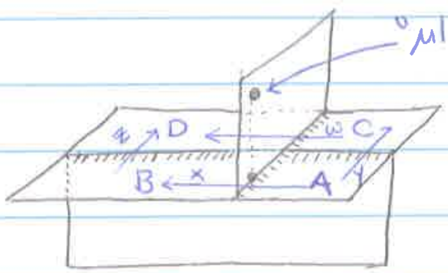
- M, a ~~f~~ manifold, $L_{con} \subset T^*M$ conical Lagr.
- Kashiwara – Schapira: Category $\mu loc(L_{con})$ of sheaves on M with microsupport in L_{con}, "microlocalize" over L_{con}

Ex. 0. $L_{con} = M \Rightarrow \mu loc(M) =$ local systems on M.

1. $M = \mathbb{R}^2$

"μlocal stalk" = Cone $(A \xrightarrow{x} B)$

$L_{con} =$



hairs = conormal directions

$$\mu loc(L_{con}) \simeq Rep(\circ \leftarrow \circ \rightarrow \circ)$$

$\mathcal{F} \in \mu loc(L_{con})$

stalks A, B, C, D $\longmapsto$ $B \xleftarrow{x} A \xrightarrow{y} C$

maps x, y, z, w

s.t. $D = \text{Cone}(A \to B \oplus C)$

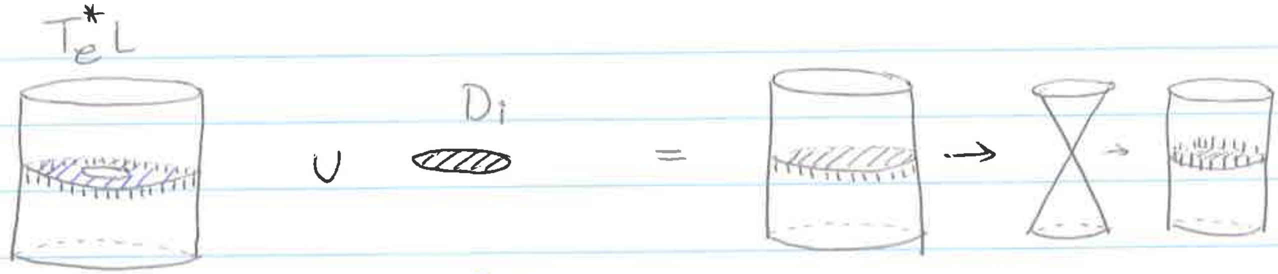
Skeletal seeds

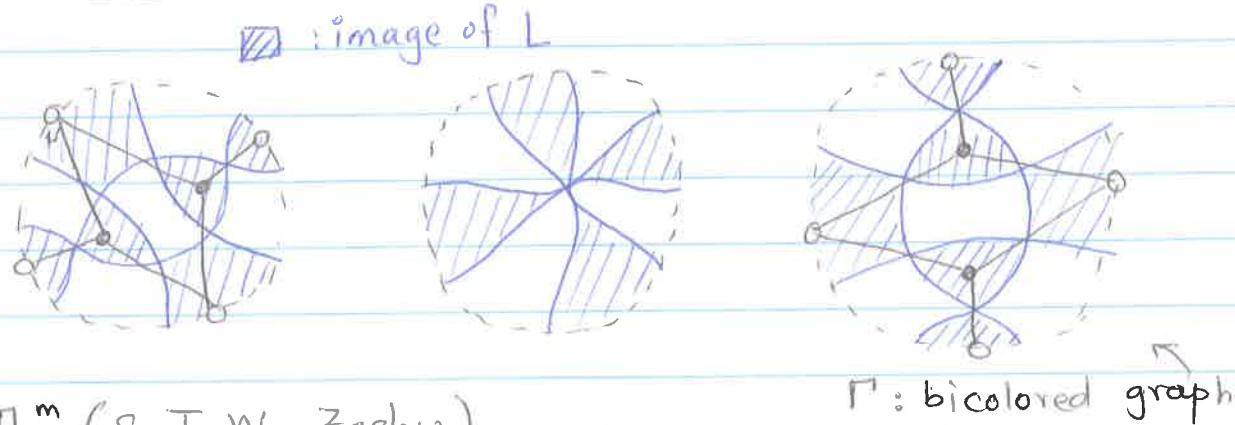
- L surface, $\mathcal{C} = \{C_i\}$ immersed co-oriented curves
  $\rightsquigarrow$ conical Lagrangian $T^*_{\mathcal{C}}L \subset T^*L$
- Attach Weinstein handles

sympl 4-mfld $W_{\mathcal{C}} := T^*L \cup \{T^*D_i\}$

retracts onto

Lag. skeletal: $S_{\mathcal{C}} := T^*_{\mathcal{C}}L \cup \{D_i\}$ ← 2-disks



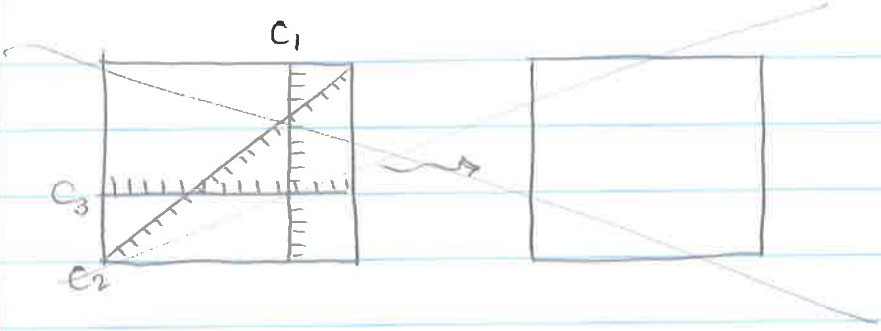


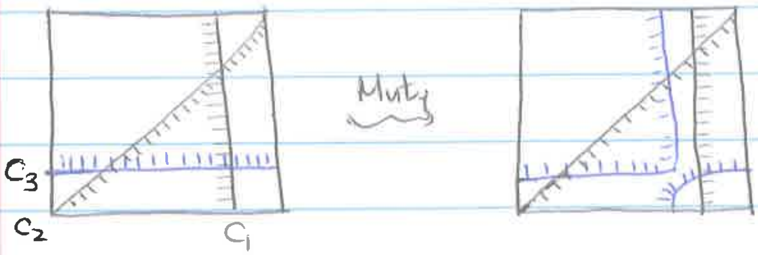
Th^m (S-T-W-Zaslow)

- Bicolored graph $\Gamma \subset \Sigma$
  $\rightsquigarrow$ exact Lagrangian $L \subset T^*\Sigma$
  $L \cap \Sigma = \Gamma \Rightarrow L \cap \Sigma = L \cup \{\text{faces of } \Gamma\}$ (Lagrangian disks)

Rmk. Larger 2-parameter family of Lagrangians:

- How to lift to an operation on S_e?
  - Define $e' := \mathrm{Mut}_b\, e$:





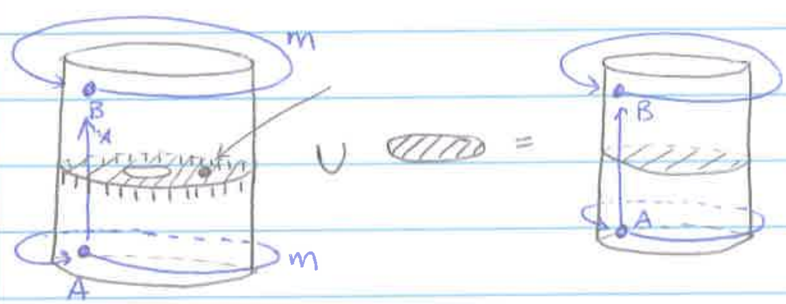
$C_i' = C_i$ reversed

$C_j' = C_j$ twisted around C_k at pos intersections

Th^m : $\exists$ symplectomorphism $\mathrm{Mut}_k : W_e \xrightarrow{\sim} W_{e'}$ and $\mathrm{Mut}_k^{-1}(L' \subset S_{e'})$ is obtained from $L \subset S_e$ by disk surgery at D_k

Rmk : The seed $(H_1(L', \mathbb{Z}), \{[C_i']\})$ is the mutation of $(H_1(L, \mathbb{Z}), \{[C_i]\})$ at $[C_k]$

- Category $\mu loc(S_e)$ of μlocal sheaves on S_e = μlocal sheaves on T_e^*L and $\{D_i\}$ together with gluing data



μloc : $\mathrm{Rep}\left(\overset{A}{\underset{m}{\circlearrowleft}} \xrightarrow{x} \overset{B}{\underset{m}{\circlearrowleft}} \right)$ $\mathrm{Rep}(\bullet)$: $\mathrm{Rep}\left(\overset{A}{\underset{m}{\circlearrowleft}} \underset{y}{\overset{x}{\rightleftarrows}} \overset{B}{\circlearrowleft} \right)$

such that $xy = 1-m$

$$\begin{tikzcd} A[1] \arrow[r, "x"] \arrow[d, "1-m"'] & B \arrow[d, "1-m"] \arrow[dl, "y"'] \\ A[1] \arrow[r, "x"'] & B \end{tikzcd}$$

trivialize the monodromy

Heuristic : $\mu loc(S_e) \simeq Fuk(W_e)$

stalks vanish on $\{D_i\}$

$$\begin{tikzcd} & \mu loc(S_e) & & Fuk(W_e) \\ & & Loc(L) \arrow[ul] \arrow[ur] & \end{tikzcd}$$

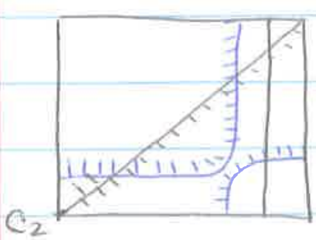
Thm : $\exists$ equivalence $Mut_k : \mu loc(S_e) \xrightarrow{\sim} \mu loc(S_{e'})$

and

$$\begin{tikzcd} Loc_1(L) \arrow[r, dashed, "\mu_k"'] \arrow[d, hook] & Loc_1(L) \arrow[d, hook] \\ \mu loc(S_e) \arrow[r, "\sim"] & \mu loc(S_{e'}) \end{tikzcd}$$

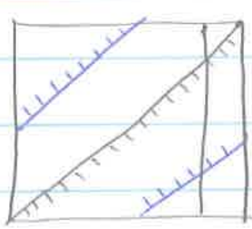
is given by $\mu_k^* z^r = z^r (1 - z^{c_k})^{\langle r, c_k \rangle}$ (cluster X-transformation)

holonomy around $\gamma \in H_1(L, \mathbb{Z})$



Mut_2

$\leftarrow$ bad



then mutate.