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Mirror symmetry for homogeneous spaces — Clelia Pech

1. Introduction
2. A Lie-theoretic mirror
3. A mirror "canonical" coordinates

1. Introduction:

$X \quad / \mathbb{C}$

(here hom. space)

mirror = Landau-Ginzburg model (LG)

$(X^\vee, W_q)$ $\quad W_q : \check{X} \times \mathbb{C}_q^* \to \mathbb{C}$

non comp Kähler; q: quantum parameter; hol.

Quantum Cohomology $\longleftrightarrow$ $\mathrm{Jac}(\check{X}, W)$ Jacobi ring

= deformation of $H^*(X, \mathbb{C})$ $\quad = \mathbb{C}[X^\vee]/(\partial W_q)$

(struct. const = counts of curves on X)

$(H^*(X,\mathbb{C}) \times \mathbb{C}_q^* \times \mathbb{C}_\hbar^*, \nabla^A)$ — ∇^A: Givental connection

$(H^N_{dR}(\check{X}, d + dW_q \wedge \_) \times \mathbb{C}_q^* \times \mathbb{C}_\hbar^*, \nabla^B)$ — ∇^B: Gauss-Manin connection

$$\nabla^A_{q\partial_q} = q\frac{d}{dq} + \frac{1}{\hbar} c_1(x) \circledast \_$$

$\circledast$: quantum product

$$\nabla^B_{q\partial_q} = q\frac{d}{dq} + \left[q\frac{dW_q}{dq} \; \_\right]$$

$$\nabla^A_{\hbar\partial_\hbar} = \hbar\frac{d}{d\hbar} + \frac{1}{\hbar} c_1(TX) * \_ + \mathrm{Gr}(\quad)$$

Gr: $\frac{1}{2}$ cohom deg

$$\nabla^B_{\hbar\partial_\hbar} = \hbar\frac{d}{d\hbar} + [W_q \_]$$

Ex : $X = \mathbb{CP}^1$ ← hyperplane class

$H^*(X) = \mathbb{C}[H]/(H^2)$

→ quantum cohom : $QH^*(X) = \mathbb{C}[H,q]/(\underbrace{H^2 = q}_{\text{thr' 2 pts, } \exists! \text{ line}})$

We know : mirror of X is

$X^\vee = \mathbb{C}^*$, $W_q = x + \frac{q}{x}$

Clearly $QH^*(X) \cong \mathrm{Jac}(X^\vee, W)$

$H \longmapsto x$

Map between vb with connection : $H \longmapsto x\,dx$

$\left(H^N_{dR}(X^\vee, d + dW_q \wedge) := \Omega^N(X^\vee) / (d + dW_q \wedge)\, \Omega^{N-1}(X^\vee) \right)$

For $X = \mathbb{CP}^N$ $\quad X^\vee = (\mathbb{C}^*)^N$

$W_q = x_1 + \cdots + x_N + \frac{q}{x_1 \cdots x_N}$

$H^i \longmapsto x_1 \cdots x_i \, dx_1 \wedge \cdots \wedge dx_i$

2. A - Lie theoretic LG model :

Assume for simplicity $G = GL_n(\mathbb{C})$, but results are valid for all semisimple alg. groups.

Notation : P = parabolic subgroup = 'block - triangular matr'

$$\underset{G}{\cap} \quad = \begin{pmatrix} \boxed{GL_{n_1}} & & * \\ & \ddots & \\ 0 & & \boxed{GL_{n_k}} \end{pmatrix}$$

B^+ = upper-triangular matrices $\subset G$
B^- = lower-triangular matrices $\subset G$
U^+ = upper-triangular with 1 on the diagonal
U^- = lower triangular with 1 on the diagonal
$W = S_n$ Weyl group
$W_P = S_{r_1} \times \cdots \times S_{r_\ell} \subset S_n$
$X = G/P$ partial flag variety

Bruhat decomposition: $G/P = \bigsqcup_{w \in W/W_P} \underbrace{B^+ w P/P}_{C_w^+}$: schubert cells

CW complex
$\Rightarrow H^*(G/P) = \bigoplus_{w \in W/W_P} \mathbb{Z}\sigma^w \qquad \sigma^w = [\overline{C_w^+}]$

Let $G^\vee$ be the Langlands dual of G
($G = \cancel{GL_n}\, SL_n \Rightarrow G^\vee = PSL_n$)
Define: $\underbrace{R^\vee_{w_0, w_P}}_{\text{open Richardson variety}} \subseteq G^\vee/B^\vee_-$ Langlands dual full flag var.

w_0 := longest element in $W = S_n$
w_P := longest element in W_P
$R^\vee_{w_0, w_P} := B^\vee_- w_0 B^\vee_- \cap B^\vee_+ w_P B^\vee_+ / B^\vee_-$ intersection of Bruhat cells.
$\downarrow$
has a cluster structure [Geiss-Leclerc-Schröer]
Need to define a potential function on $R^\vee$: $\longrightarrow$

$$R^\vee_{w_0, w_P} \cong U_+^\vee (T^\vee)^{w_P} \dot{w}_0 \dot{w}_P U_-^\vee \cap B_-^\vee w_0$$

$$F : R^\vee_{w_0, w_P} \longrightarrow \mathbb{C}$$

$$\underbrace{u_1}_{\in U^+} t \dot{w}_0 \dot{w}_P \underbrace{\bar{u}_2}_{\in U^-} \longmapsto \sum_{i=1}^{N-1} \underbrace{e_i^*(u_1)}_{\substack{\text{coeff } (i, i+1) \\ \text{of } u_1}} + \sum_{i=1}^{N-1} \underbrace{(f_i^*(\bar{u}_2))}_{\substack{\text{coeff } (i+1, i) \\ \text{of } \bar{u}_2}}$$

Thm [Rietsch 2008]

$$QH^*(G/P)_{loc} \cong \mathrm{Jac}(R^\vee_{w_0, w_P}, F)$$

Ex. $X = \mathbb{CP}^1$ $G = SL_2$

$$R^\vee_{w_0, w_P} = \left\{ \underbrace{\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix}}_{M} \;\middle|\; zw = q \right\} \cong \mathbb{C}^*$$

$$F : M(z, w) \longmapsto e_1^* \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} + f_1^* \begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix} = z + w = z + \frac{q}{z}$$

$\rightsquigarrow$ recovers the usual LG model

Summary: For $X = G/P$, get

$$\begin{array}{c} \text{log CY, smooth affine} \\ R^\vee_{w_0, w_P} \subseteq G^\vee / B_-^\vee \\ \downarrow F \\ \mathbb{C} \end{array}$$

3. $\pi : G^\vee / B_-^\vee \longrightarrow G^\vee / P^\vee$

$X^\vee = \pi(R^\vee_{w_0, w_P}) \cong R^\vee_{w_0, w_P}$, $W_q = \pi_* F$

(Knutson-Lam-Speyer)

$$\begin{array}{ccc} X = G/P & \longleftrightarrow & X^\vee \subset G^\vee / P^\vee \\ & & \downarrow W_q \\ & & \mathbb{C} \end{array}$$

$$G^\vee/P^\vee \hookrightarrow \underbrace{\mathbb{P}(V_{P^\vee}^{G^\vee})^*}_{\text{Some rep of } G^\vee} \underset{\text{geom Satake}}{\cong} IH^*(\underbrace{Gr_{G^\vee}^{P^\vee}}_{\text{some cycle in the affine grass } Gr_G})$$

When P is cominuscule (e.g. $X = Gr(k,n)$ $X =$ quadratic
$X =$ Lag. grass.)
we have:

$$\mathbb{P}(V_{P^\vee}^{G^\vee})^* \cong \mathbb{P}(H^*(G/P))$$

If G/P is cominuscule, $\check{X}$ has coordinates in 1-1 correspondance with Schubert classes on X.

Ex: $X = \mathbb{CP}^1$

Schubert classes are $1, H$ $\longleftrightarrow$ $z + \frac{q}{z}$ (→ coordinate)

Next time: Ex. $Gr(k,n)$ [Marsh - Rietsch]
[Rietsch - Williams]

- Quadrics
- Lagrangian Grass.