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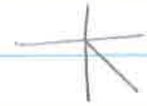
Scattering diagrams from stability conditions

— Tom Sutherland

Q : acyclic quiver with n vertices $\{1, \dots, n\}$

Construct a scattering diagram associated to Q by studying the category of representations of Q.

Example will be $Q : \bullet \longrightarrow \bullet$ $\rightsquigarrow$ 

Let $\mathcal{A}$ be the cat. of reps of Q.

$\mathcal{A}$ is an abelian category.

Define $K(\mathcal{A})$: Groth. group of $\mathcal{A}$ is the free ab. gp on isom. classes of objects of $\mathcal{A}$ modulo

$[X] = [A] + [B]$ if $\exists$ $0 \to A \to X \to B \to 0$

For $Q : \bullet \longrightarrow \bullet$,

$K(\mathcal{A}) \cong \mathbb{Z}^2$

In general $K(\mathcal{A}) \cong \mathbb{Z}^n$ (for acyclic quiver w/ n vertices)

$K(\mathcal{A})$ is generated by the classes of simple objects of $\mathcal{A}$: $S_1, S_2, \dots, S_n$

S_i : representation with one-dim vector space at i.

some field.

For $Q : \bullet \longrightarrow \bullet$, $S_1 : k \to 0$, $S_2 : 0 \to k$

We can define the Euler form $\chi_\mathcal{A}$ on $K(\mathcal{A})$

$\chi_\mathcal{A}([X], [Y]) = \dim \mathrm{Hom}(X, Y) - \dim \mathrm{Ext}^1(X, Y)$

($\mathcal{A}$ is heredetory as Q is acyclic)

Define $\langle \, , \, \rangle$ to be anti-symmetrization of $\chi_\mathcal{A}$.

In our ex : $K(\mathcal{A}) \cong \mathbb{Z}^2 = \mathbb{Z}[S_1] \oplus \mathbb{Z}[S_2]$

$\langle [S_1], [S_2] \rangle = -1$

$\therefore$ form $= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Say an object X of $\mathcal{A}$ is $\underline{\Theta\text{-stable}}$ w.r.t $\Theta \in \mathrm{Hom}(K(\mathcal{A}), \mathbb{R})$

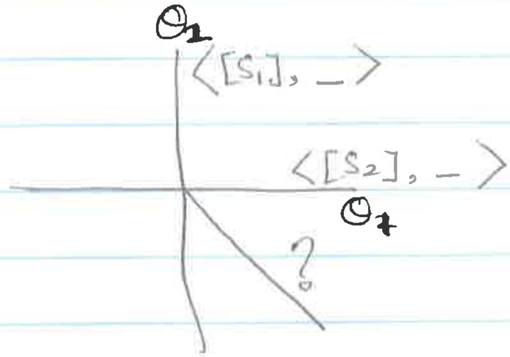
dual to the lattice

if ① $\Theta([X]) = 0$

② For any sub object $0 \neq S \subsetneq X$

$\Theta([S]) < 0$.

The support of scattering diag is those Θ for which $\exists$ Θ-stable object.



$[S_2]$ is the stable object for Θ_1

& $[S_1]$ for Θ_2 ,

$[E]$ for the ? .

There's a SES in $\mathcal{A}$, $0 \longrightarrow S_2 \longrightarrow E \longrightarrow S_1 \longrightarrow 0$

which gives an indecomp E.

$$\begin{array}{ccccc} 0 & \longrightarrow & k & \longrightarrow & k \\ \downarrow & & \downarrow & & \downarrow \\ k & & k & & 0 \end{array}$$

$[E] = [S_1] + [S_2]$

S_2 proper nontrivial subrep of E

& $\Theta([S_2]) < 0$

$\therefore$ Θ has to be a positive multiple of $(1, -1)$.

☆ A potential W on Q is some finite sum of finite cycles on Q

Q acyclic $\Rightarrow$ $W = 0$

Consider Jacobi algebra of (Q, W) :

$J(Q,W) = kQ/\text{relations from } W$ $\quad$ kQ: path alg of Q.

Can apply above methods to the cat,

$\mathcal{A} = \text{mod}(J(Q,W))$

There is a notion of mutation of quiver w/ potential (Q, W) at a vertex k of Q

$(Q,W) \rightsquigarrow \mu_k(Q,W)$

If W is non-degenerate, then the underlying quiver of $\mu_k(Q,W)$ agrees with the usual mutation.

Ex. $\bullet \longrightarrow \boxed{\circ} \longrightarrow \bullet$ $\quad \overset{\mu}{\rightsquigarrow} \quad$
$$\begin{tikzcd} \circ \arrow[rr] & & \circ \arrow[dl] \\ & \circ \arrow[ul] & \end{tikzcd}$$

$W = 0$ $\qquad\qquad$ $W = \nwarrow\!\rightarrow\!\swarrow$

Given (Q, W), $\exists$ a triangulated category

$D_{(Q,W)} = D^b(\text{Mod}\,\Gamma(Q,W))$ $\quad$ (Ginzburg's dg alg)

heart of bdd t-struct. $\quad \cup\!\!|$ $\qquad\qquad$ & $H^0(\Gamma(Q,W)) = J(Q,W)$

$\mathcal{A} = \text{Mod}(J(Q,W))$

$D_{(Q,W)}$ is CY-3 category, i.e. $\text{Hom}^i_D(X,Y) \cong \text{Hom}^{3-i}_D(Y,X)^\vee$

In particular, χ_D on $K(D)$ is skew-symm

Fix (Q,W) & vertex k, $\exists$ a pair $\phi^\pm$ of equivalences

$\phi^\pm : D^b(\text{Mod}(\Gamma(Q,W))) \longrightarrow D^b(\text{mod}(\Gamma\,\mu_k(Q,W)))$

$\phi^\pm(\text{Mod}\,J(Q,W)) = \mu_k^\pm(\text{Mod}\,J(\mu_k(Q,W))$

left/right tilt of the heart wrt tortion pair for which T or f is $\{S_k^{\oplus n}\}$

heart $\leftarrow$ central charge

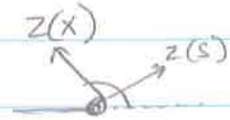
Stability conditions on $D(Q,W)$: pair (A, Z)

$Z: K(D) \to \mathbb{C}$

s.t. for any object $X \in \mathcal{A}$, $Z([X]) \subseteq$

Say an object X is stable if $\forall\, 0 \subset S \subsetneq X$

$\arg(Z(S)) < \arg(Z(X))$



Ex. $\mathcal{A} = \mathrm{mod}(J(Q,W))$

Given $\Theta \in \mathrm{Hom}(K(\mathcal{A}), \mathbb{R})$

$Z(X) = -\Theta(X) + i\dim(X)$

X is ~~Z~~ Z stable w/ $\arg(X) = \frac{\pi}{2} \iff \Theta$-stable