Talia Fernos

Regular isometries of CAT(0) cube complexes are plentiful.

w/ Lécureux + Mathéus. (A)

w/ Chatterji + Iozzi (B)

Caprace + Sageev :

$\Gamma \longrightarrow \mathrm{Aut}(X)$ (so $\Gamma \curvearrowright X$).

X : fin. dim. CAT(0) c.c. and

Γ action is irreducible, essential, non-elementary

$\Rightarrow$ $\exists$ rank-1 isometry in image

X : fin. dim'l, (not nec. loc. compact

def$^{\underline{n}}$: $\Gamma \curvearrowright X$ is NON-ELEMENTARY

if DNE a fin. orbit in $X \cup \partial X$.

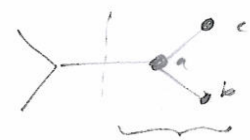
visual boundary

notation :

$\mathcal{H}$ = { half-spaces h }

where h = { all vertices on one side of a hyperplane }.

eg.



X = tree h = { a, b, c }.

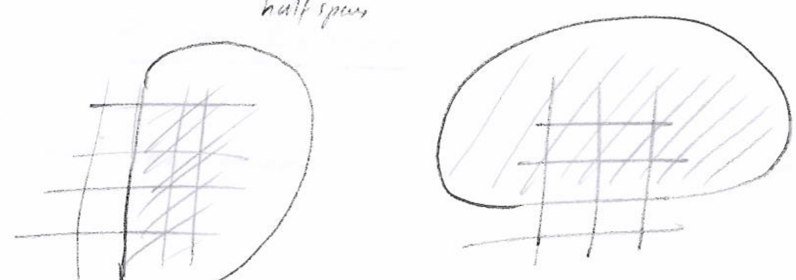
in product regions :

$X = X_1 \times X_2$

set of half spaces

$\mathcal{H}(X) = \mathcal{H}(X_1) \sqcup \mathcal{H}(X_2)$

vertical half spaces



$h_1 \times X_2$ $X_1 \times h_2$

Aut (X) = cubical isometries
in both CAT(0)
and edge metric.

Important properties:

$\Gamma \curvearrowright X$ is essential
if every half space is
crossed by a hyperbolic element

def^n (Behrstock + Charney)
two walls $h, k \in \mathcal{H}$ are

(SS) STRONGLY SEPARATED if DNE hyperplane
crossing both h, k

and

Forbid

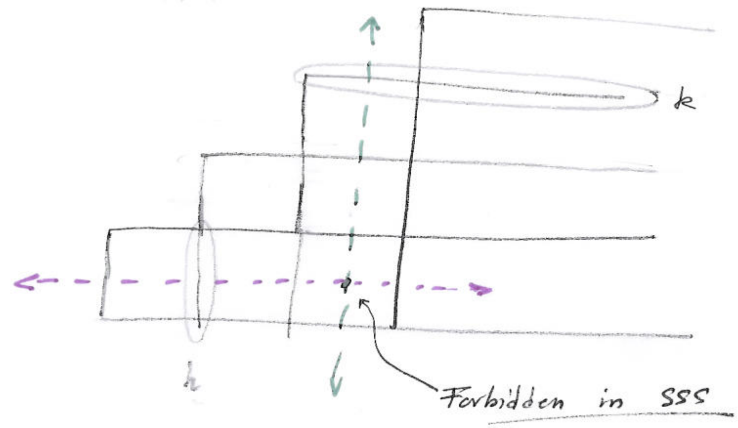
(SSS) SUPER-strongly separated if any
hyperplane crossing h may NOT cross
a hyperplane that crosses k.

Forbid

Rk: distance in the contact graph is > 2

N.B. the staircase (obstacle to HHS
and unif loc. fin.)

is (SS), but not (SSS)



Caprace + Sageev

X: essential + non-elementary
(Aut (X) has these properties)

X: irreducible

$\Leftrightarrow$ $\exists$ pair of SS half planes

$\Leftrightarrow$ $\exists$ pair of SSS half planes.

Behavior of elements in $Aut(X)$

- $\gamma \curvearrowright h$ acts as a FLIP

  if

- $\gamma \curvearrowright h$ acts as a SKEWER

  if

Caprace + Sageev:

$\Gamma \curvearrowright X$ is essential + non-elementary

$\Rightarrow$ • $\forall\, h, k \in \mathcal{H}$ s.t. $h \subset k$.

then $h \subset k$ in ∂X (double skewering)

• if h, k are s.s.

then $\exists$ rank-1 element γ and $h \subset k$ in ∂X

Corollary

If Γ is essential + non elementary

X : irreducible

then rank-1 elements exist.

X : essential and $X = X_1 \times X_2$

$$Aut(X_1) \times Aut(X_2) \leq Aut(X)$$

def^n a REGULAR ELEMENT in

$$Aut(X_1) \times \ldots \times Aut(X_n)$$

is rank-1 on each factor X_i.

Main Thm (A)

$\Gamma \curvearrowright X$ non-elementary + essential.

really just need fin. dim'l ?

$\Rightarrow$ regular elements exist.

past special cases

Caprace + Sageev: X : irred or $\Gamma < Aut(X)$ a lattice.

Caprace + Zadnik: X "nice" CAT(0) space

Γ : cocompact lattice.

idea: push forward random walks on Γ to X.

Pick $\mu \in \mathrm{Prob}(\Gamma)$ symmetric

measure

s.t. $\mathrm{supp}(\mu)$ generates Γ.

$Z_n = g_1 \cdots g_n$ a random walk of length n.

g_i are i.i.d with μ.

associated to this probability measure

have B = Poisson Boundary (introduced by Furstenberg, but not Furstenberg boundary)

= "limits" of these random walks

Bader + Furman : "super super duper duper" Ergodicity.

strong "double" ergodicity

Zimmer ammenability :

$\Gamma \curvearrowright X \implies \exists$ Γ-equivariant measure on X given by $B \longrightarrow \mathrm{Prob}(\overline{X})$

Fernos :

$$\boxed{\begin{array}{l} \Gamma \curvearrowright X \text{ essential + non-elementary} \\ \implies \exists \ B \longrightarrow \partial_r X \subset \partial X \\ \quad \Gamma\text{-equivariant} \end{array}}$$

Roller?

Main Thm A 2

$\exists!$ Γ-equiv. measurable $B \longrightarrow \overline{X}$

corollary : $\exists!$ "stationary" measure on $\overline{X}$ given by pushing forward the one on B.

ν

Main Thm B

$\nu(h) > 0 \quad \forall h \in \mathcal{H}$

$\Downarrow$

$\nu(\partial_r X) = 1$

where stationary means on average with μ measure is ν-invariant.

Main Thm A 3

$\Gamma \curvearrowright X$ "nice" + preserves each factor.

almost every random walk (Z_n)

$Z_n \cdot \theta \longrightarrow \xi \in \partial_r X$

Lemma (A)

X: rmd, Γ: nice, (z_n): generic

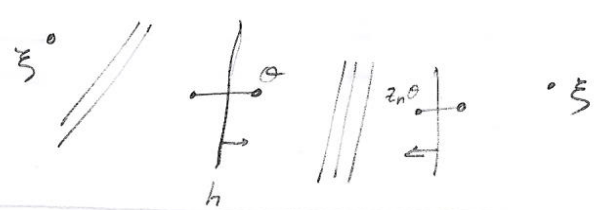
s.t. $z_n\theta \longrightarrow \xi \in \partial_r X$

if $h \in \mathcal{h}$, $\xi \in h$

then $\exists\, N$ s.t. $\forall\, n > N$.

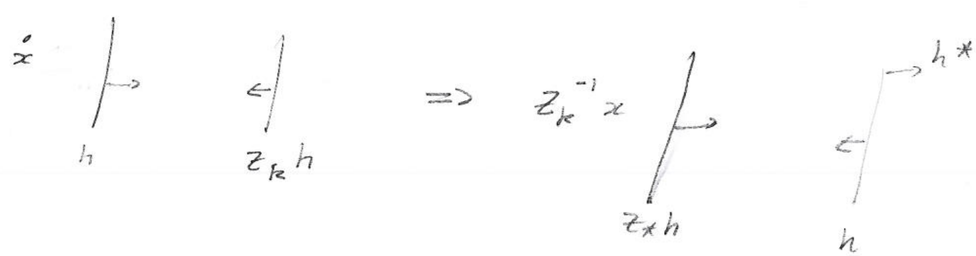
(1)

or (2)



proof of Thm (A) 3

(sum) $\leq$ $\#\{k \leq n$:  $\} + N$



$\mathbb{1}_{h^*}(z_k^{-1} x) = 0$

h: both closed + open $\Rightarrow$ $\mathbb{1}_h$ continuous.

Benoist + Quint

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \mathbb{1}_{h^*}(z_k^{-1}X) = \nu(h) > 0$$

$$\lim \frac{1}{n}\{k : z_k \text{ rank-}1\} = 1$$