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Hierarchically hyperbolic structures on cube complexes and applications.

w/ M. Hagen + J. Behrstock.

Examples: of HHS's (hierarchically hyperbolic spaces)

- hyperbolic spaces
- hyperbolic spaces relative to HHS.
- X: CAT(0) c.c. s.t. $G \curvearrowright X$ is a geometric action.
- MCG's
- Teichmüller space either · Teich-metric · W-P metric.
- π_1 (non-geometric 3-mflds)

Idea (Masur + Minsky study of MCG).

Study geometry of X to study of hyperbolic spaces

eg. $X = MCG \curvearrowright \mathcal{C}$ (subsurfaces)

⚠ Warning! the same space may have several HHS structures.

X = CAT(0) c.c. and structure depends on "Factor systems"

eg. $X = \tilde{S}_\Gamma$: Salvetti (raag)

$\mathcal{F} = \{ g \tilde{S}_\Lambda \}_{g \in A(\Gamma)}$ $\Lambda \subset \Gamma$ subgraph.

factor system

$\exists$ collection of convex subcomplexes

$V \subset X$ in a factor system

provided

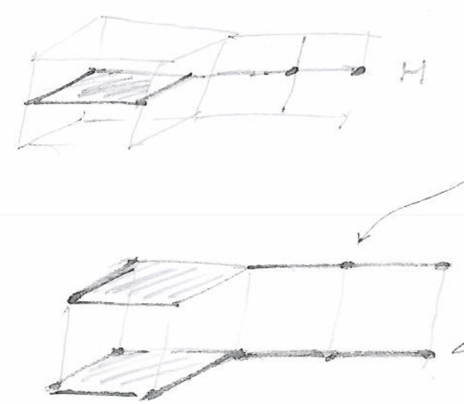
(1) $X \in \mathcal{F}$, and all combinatorial hyperplanes $\in \mathcal{F}$

(2) $\mathcal{F}$ is closed under projections

as subcomplexes of X

* Gates *

(3) $\mathcal{F}$ is uniformly loc. finite.



Hagen + Susse

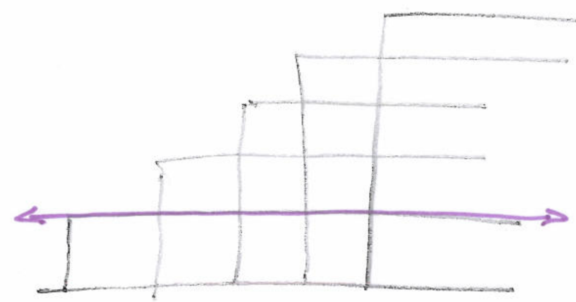
if $G \curvearrowright X$ is geometric. (proper + cocompact.)

then X has a factor system.

in particular unif. loc. fin.

troublemaker/ ~~poison~~ subcomplex: "staircase"

Proposition: $C(X)$ is a quasi tree



not uniformly loc. finite. projections of other comb. hyperplanes onto this one.

Rk existence of a factor system

$\Rightarrow$ $\exists$ minimal such system

Def^n: $C_{\mathcal{F}}(X) = C(X) = X$ with all proper factors $F' \subsetneq X$ coned off

similarly for any $F \in \mathcal{F}$

$C_{\mathcal{F}}(F) = C(F) = F$ with ...

Rk: F convex $\subset X \Rightarrow F$ is CAT(0) sub-C.C.

we call $\mathcal{F}$: the factored contact graph

only cone sub-complexes of F

$$\begin{tikzcd} \pi_F : X \arrow[rr] \arrow[dr, "\text{proj.}"'] & & C(F) \\ & F \arrow[ur] & \end{tikzcd}$$

$\bar{\mathcal{F}}$:= set of representatives of parallelism class.

$F \parallel F'$ $\Longleftrightarrow$ have a common crossing / transversal hyperplane.

eg. in case when $X = \tilde{S}_\Gamma$

$$g\tilde{S}_\Lambda \parallel h\tilde{S}_\Lambda \Longleftrightarrow [gh^{-1}, A(\Lambda)] = \underset{\text{trivial}}{\mathbb{1}}$$

Distance Formula Thm: $\forall L \gg 0$ $\exists K, C$ s.t.

$$\boxed{d_X(x,y) \underset{K,C}{\asymp} \sum_{F \in \bar{\mathcal{F}}} \left[d_{C(F)}(\pi_F(x), \pi_F(y)) \right]_L}$$

compare to subsurface proj.
distance formula in cube complex.

cutoff
(ignore values below L)

Moreover, $\forall x, y$ $\exists$ geodesic $\gamma : x \rightsquigarrow y$ s.t. $(\pi_F \circ \gamma)$ is an unparametrized quasi-geodesic $\forall F$.

γ is called the HIERARCHY PATH

wording borrowed from Masur+Minsky.

General HHS when $X \neq$ CAT(0) c.c.

- X : metric space.
- $\{C(Y)\}_{Y \in \mathcal{F}}$ all (uniform) δ-hyp.
- $\{\pi_Y : X \to C(Y)\}$ projection maps.
- (weak distance formula)
- (partial realization)
  - easier to verify than full versions.
- Bounded geodesic image.
- other "smaller" conditions.

Rk: Y need not be hyperbolic.

consequences

- (not weak) distance formula
- hierarchy paths.
- analog of Realization Thm.
  (in MCG, partial marking can be can be extended to full marking)

Applications:

Thm: $G \curvearrowright X$ geometric.

$\mathfrak{F}$ = G-invariant factor system.

$\Rightarrow$ $G \curvearrowright C(X)$ (nec. not proper) is acylindrical

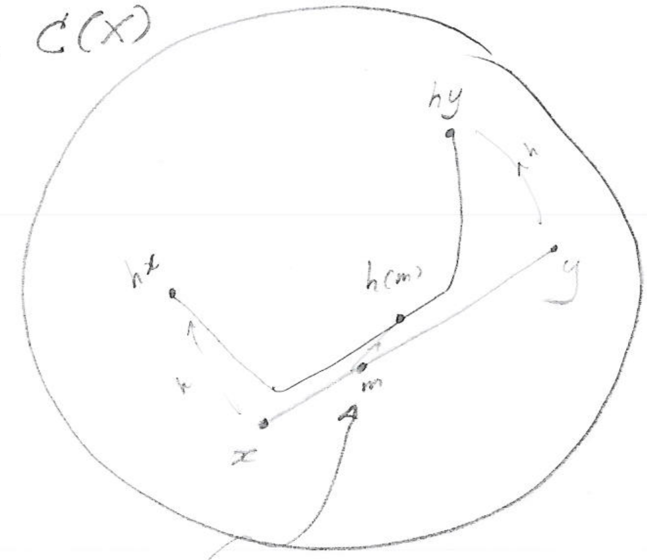
because of coning

Rk: $C(X)$ is not nec. ∞-diameter so do NOT have acylindrical hyperbolicity.

idea for WPD:

(slightly weaker than acylindrical

in $C(X)$



π_X (hierarchy path)

Goal: bound $d_X(m, hm)$

using DF

Assume $d_{C(F)}(\pi_F(x), \pi_F(y)) \asymp 0$

$\forall F \subsetneq X$.

Bounded Geod image

$\pi_F(m) \asymp \pi_F(x) \asymp \pi_F(hx) \asymp \pi_F(hm)$

$\forall F \subsetneq X$.

Currently, no quasi-isometric rigidity.

HHS $\Rightarrow$ quadratic isoperimetric.

$\Rightarrow$ no Out(F_n)