THE GEOMETRY OF CAT(0) SPACES

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ABSTRACT. The talk will begin with a brief history of CAT(0) geometry, including some long-standing open problems. Then I will discuss more recent developments and areas of current interest, including the theory of CAT(0) cube complexes and the interplay between CAT(0) geometry and hyperbolic geometry.

Outline

I. Dawn of CAT(0) geometry

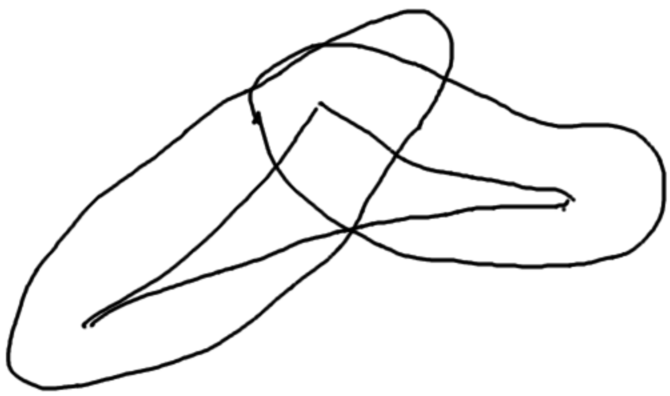
II. Invasion of the CAT(0) cube complexes

III. Marriage of hyperbolic and CAT(0) geometry

1. DAWN OF CAT(0) GEOMETRY

Gromov 1987: two notions of "curvature"

- coarse: δ-hyperbolic



triangles are δ-thin

- fine: CAT(K) space, $K \leq 0$

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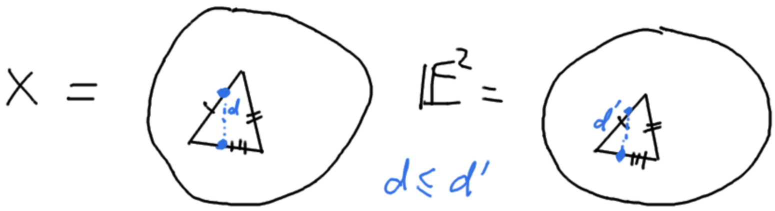
all triangles in X are "thin" compared to triangles in $\mathbb{E}^2$ ($K = 0$) or $\mathbb{H}^2$ ($K = -1$).

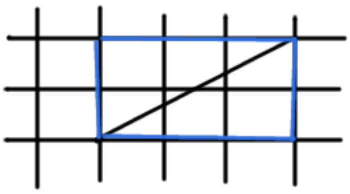


Theorem. *If X is locally CAT(K) and simply connected then X is CAT(K).*

CAT(-1) $\implies$ δ-hyperbolic, and CAT(-1) $\implies$ CAT(0).

Being hyperbolic (for some δ) is a quasi-isometry invariant, whereas being CAT(0) is *not* a quasi-isometry invariant.

(e.g. $\mathbb{Z}^2$ vs $\mathbb{R}^2$)



Definition. A group G is *hyperbolic* (respectively *CAT(0)*) if G acts geometrically (i.e. properly, cocompactly by isometries) on a hyperbolic (respectively CAT(0)) space.

Theorem (Svarc–Milnor Theorem). *If* $G \overset{geom}{\curvearrowright} X$ *then* $G \sim_{QI} X$.

$\rightsquigarrow$ G is hyperbolic $\iff$ some (hence any) Cayley graph of G is hyperbolic.

How to show G is CAT(0)?

$\mathrm{Cay}(G)$ is never CAT(0) unless $G =$ free group.

Must construct some CAT(0) space X and geometric action $G \curvearrowright X$.

Early questions.

- Which groups are CAT(0)?

  (e.g. Moussong: *all* Coxeter groups are CAT(0).)

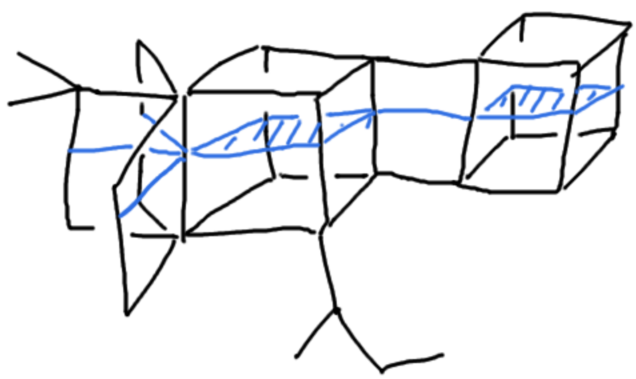
- How to effectively construct CAT(0) spaces?
- What (algebraic) properties do CAT(0) groups satisfy?

Still open.

- Are Artin groups CAT(0)? Are braid groups CAT(0)?
- Is every hyperbolic group CAT(0)?
- If $H < G$ finite index, H CAT(0) $\overset{?}{\Longrightarrow}$ G CAT(0)?
- Is the isomorphism problem solvable?
- Are all CAT(0) groups automatic?
- Do CAT(0) groups satisfy a Tits Alternative?

2. Invasion of the CAT(0) cube complexes

Definition. A *cube complex* is a geodesic metric space X formed by gluing Euclidean cubes ($[0,1]^k$) by isometries along faces.

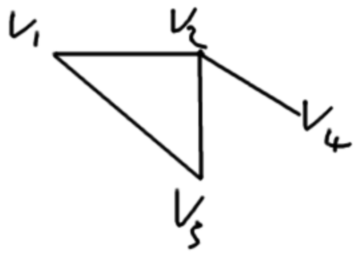


(note: blue indicates hyperplanes, introduced later)

Gromov: A cube complex is locally CAT(0) $\iff$ links of vertices are flag complexes $\iff$ X has "no empty corners".

This condition is easy to check!

Example (Right-angled Artin group). Γ = finite graph, $V(\Gamma) = \{v_1, \ldots, v_n\}$

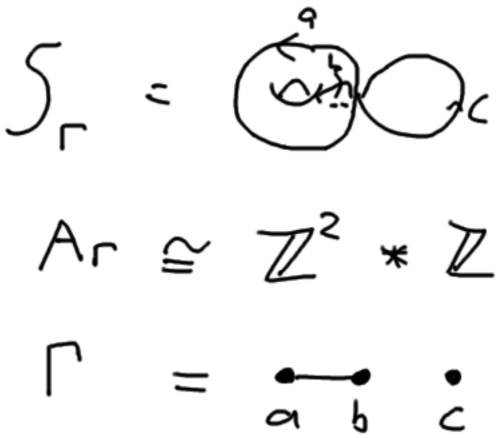


$A_\Gamma = \langle\, v_1, \ldots, v_n \mid v_i v_j = v_j v_i \text{ if } v_i - v_j \in E(\Gamma) \,\rangle$

Salvetti complex:

$T^n = n$-torus

$T^n \supset S_\Gamma = \bigcup$ faces labelled by commuting sets of generators



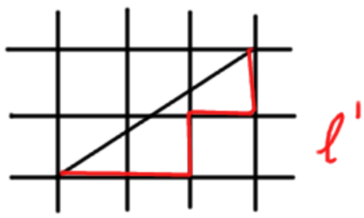
S_Γ is always locally CAT(0)

CAT(0) cube complexes (CCC's) come equipped with combinatorial as well as geometric structure.

- l^1-metric (metric on $X^{(1)}$)

  (QI to CAT(0) metric)

- hyperplane structure (indicated in blue on above example of cube complex)



Using this one can show for $G \overset{geom}{\curvearrowright} X = \text{CCC}$

- G is bi-automatic (Niblo–Reeves)
- G satisfies Tits Alternative (Sageev–Wise)
- If X/G is "special" and G is torsion-free, then $G \hookrightarrow A_\Gamma$ QI-embedding.

3. Marriage of hyperbolic and CAT(0) geometry

Theme: generalize theorems / techniques from hyperbolic geometry / groups to more general groups / spaces.

- relatively hyperbolic groups (Gromov, Farb, Bowditch, ...)
- acylindrically hyperbolic groups $G \curvearrowright X =$ hyperbolic acylindrically (Osin, ...)
- group $\hookrightarrow$ finite product of hyperbolic groups
- hierarchically hyperbolic groups (Behrstock–Hagen–Sisto)
- groups with non-trivial Morse boundary

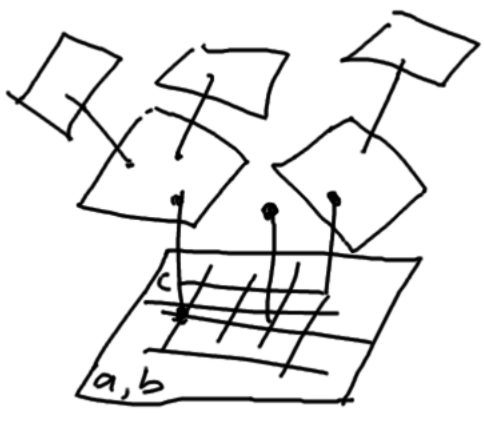
Morse boundaries

CAT(0) spaces often display some hyperbolic behavior.

Example. $A_\Gamma = \mathbb{Z}^2 * \mathbb{Z}$ generated by a, b, c.

$S_\Gamma = T^2 \vee S^1$.

$X = \widetilde{S_\Gamma} =$ "tree of flats"



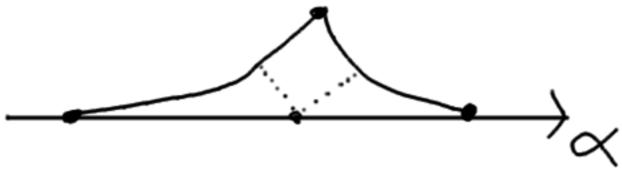
$G \overset{geom}{\curvearrowright} X = \text{CAT}(0)$, every $g \in G$ infinite order has an axis α_g and $min(g) = \{x \mid d(x, gx) \text{ is minimal}\} = \alpha_g \times Y$.

e.g.: $min(a)$:

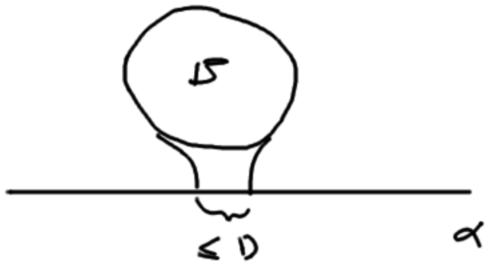
Properties of geodesics α in *hyperbolic* space.

(1) rank one: α does not bound a half-flat

(2) thin triangles: $\exists\, \delta$ such that triangles based on α are δ-thin



(3) contracting: $\forall$ balls B with $B \cap \alpha = \emptyset$, $\pi_\alpha(B)$ has diameter $\leq D$



(4) Morse: Given $(\lambda, \epsilon), \exists\, N = N(\lambda, \epsilon)$ such that every (λ, ϵ)-quasi geodesic β with endpoints on α stays in N-neighborhood of α

Theorem. *If X is proper, CAT(0) space then a geodesic (ray or line) satisfies (2) $\iff$ (3) $\iff$ (4).*

If α is periodic ($\alpha = \alpha_g$), then (1) $\iff$ (2) and so on.

Example. $G = A_\Gamma$ as above, $\alpha = aca^2ca^3ca^4c\dots$

Definition. The *Morse* boundary of a proper CAT(0) space is

$$\partial_* X = \{\alpha : [0,\infty) \to X \,|\, \text{Morse rays}\} /\!\sim$$

where $\alpha \sim \beta$ if $d_{Haus}(\alpha, \beta) < \infty$.

C.–Sultan, Cordes generalized CAT(0) to geodesic spaces.

$\partial_* X = \varinjlim \partial^N X$ (N-Morse)

Theorem (C.–Sultan, Cordes)**.** *X, Y proper geodesic metric spaces. If $f : X \to Y$ is a quasi-isometry, then f induces a homeomorphism $\partial_* f : \partial_* X \to \partial_* Y$.*

Corollary. *$\partial_* G$ is well-defined.*

Problems.

- Use $\partial_* G$ to distinguish QI-classes of groups.

- Use $\partial_* G$ to prove rigidity theorems.
- How are topological properties of $\partial_* G$ reflected in algebraic properties of G?
- Study geodesic flows on $\partial_* X$.
- Study $\partial_* X$ for CCC's. How is it related to Roller boundary and simplicial boundary.