Talk 1

$= \langle S \mid R' \rangle$

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$\Gamma = \langle S \mid R \rangle$ $|S| < \infty$

$d_S(g,h) = \|g^{-1}h\|_S$ ~ min word length in S (left inv.)

$\frac{1}{K} d_S(g,h) \le d_{S'}(g,h) \le K d_S(g,h)$

id: $(G, d_S) \longrightarrow (G, d_{S'})$ is biLipschitz

① Study gps up to biLip equiv.

Con: stuck in the world of discrete metric spaces.

② Study gps up to Quasi-isometric equivalence

Defn $f: X \longrightarrow Y$ is a (K, C) QI

if

① $-C + \frac{1}{K} d(x,y) \le d(f(x), f(y)) \le K d(x,y) + C$

② $\mathrm{nbhd}_C(f(X)) = Y$

e.g. $\mathbb{Z}$

$\mathbb{R}$

Check if $f: X \to Y$ is QI then $\exists$ $\bar{f}: Y \to X$ QI (coarse inverse)

s.t. $f \circ \bar{f}$ bdd dist from Id

Warning ① & ② are not the same in general

- If $K=1$, $C=0$ this is isometric equiv
- If $C=0$ biLip
- ~~bdd sets QI to a point~~
- ~~f QI bdd dist from g $\Rightarrow$ g QI~~

QI Rigidity

- Classify all groups QI to X (~~or a family~~)
  (X a gp)

~~QI invariant~~

e.g – ~~finite index~~ $[\Gamma:\Lambda]<\infty$ then $\Gamma \underset{QI}{\sim} \Lambda$

– ~~finite~~ also extensions of & by finite gps give ~~QI~~ gps.

e.g $X=\mathbb{R}^n$, all free gps QI to each other

- QI invariants: If $\Gamma \underset{QI}{\sim} \Gamma'$ & Γ has property BLAH ~~have property BLA~~ then so does Γ'

(called geometric properties)

[• # of ends • splittings • growth]
• amenability
• hyperbolicity
  – boundary type

google

QI Rigidity through Quasi-actions

Fund. Thm of GGT: If $\Gamma \curvearrowright X$ proper geod. metric space

– cocompactly (X/Γ cmpt) geometrically / isom (by isometries)
– prop. disc. $\forall K$ cmpt $\#\{g \mid gK \cap K \neq \emptyset\} < \infty$

then $(\Gamma, d_S) \underset{QI}{\sim} X$ $\quad \Gamma \xrightarrow{\varphi} \Gamma \cdot x_0 \subseteq X$

e.g. $\Gamma, \Lambda \curvearrowright X$ geom $\quad \Gamma \underset{QI}{\sim} \Lambda$

[Any $\Gamma \subseteq \mathrm{Isom}(X)$ ~~coco~~ uniform (cocompact) lattices are QI to X]

• View $\Gamma \subseteq \mathrm{Isom}(X)$

Partial Converse

$\boxed{Q}$ If $\Lambda \overset{QI}{\sim} X$ does $\underset{\text{geom.}}{\Lambda \curvearrowright X}$?

$\boxed{A}$ Λ "quasi-acts" on X ~~(coboundedly)~~ (properly & coboundedly)

$\Lambda \underset{\bar{f}}{\overset{f}{\rightleftarrows}} X \qquad g \in \Lambda \longmapsto \underbrace{f \circ L_g \circ \bar{f}}_{A_g} : X \to X$ a QI

- $A_g \circ A_h \quad =_B \quad A_{gh}$

$f \circ L_g \underbrace{\bar{f} \circ f}_{=_B \, Id} \circ L_h \bar{f} \qquad f \circ L_{gh} \circ \bar{f}$

$\Rightarrow$

- $A_{g^{-1}} = \bar{A_g}$
- ~~action is cobounded.~~ & (proper)
- all (K, C) QI for fixed K & C.

So $\Lambda \xrightarrow{\varphi} QI(X) = \{ f : X \to X \ \text{QI} \} / \sim$ bdd dist

- $\varphi(\Lambda) \subseteq QI(X)$ is 'uniform'
- ~~If X is 'nice' then~~ Want $|\ker(\varphi)| < \infty$
- Note $\mathrm{Isom}(X)_{/\sim} \subseteq QI(X)$ (e.g $X = \mathbb{R}$)

Goal $\quad \Lambda \to \mathrm{Isom}(X') \quad (X' = X$ or not$)$

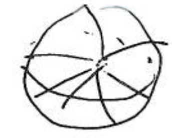
Strongly QI rigid X: $\quad QI(X) = \mathrm{Isom}(X)$

If $\Lambda \underset{QI}{\sim} X$ then $\Lambda \xrightarrow{\varphi} QI(X) = \mathrm{Isom}(X)$

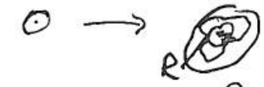
eg $\quad X$ symmetric space rank ≥ 2

$\mathrm{Isom}(X)$ s.s Lie gp

Case $X = \mathbb{H}^{n+1}$ $\quad \partial \mathbb{H}^{n+1} \simeq S^n$



$$QI(\mathbb{H}^{n+1}) \equiv QConf(S^n)$$
$$\cup\!\!\!\mid$$
$$Isom(\mathbb{H}^{n+1}) \equiv Conf(S^n)$$



$\frac{R}{r} \leq K$

IF $\Lambda \underset{QI}{\simeq} \mathbb{H}^{n+1}$

$$\Lambda \xrightarrow{\varphi} QI(\mathbb{H}^{n+1}) \equiv QConf(S^n)$$

$\varphi(\Lambda)$ • is uniform — same K

• acts cocompactly on distinct triples of S^n

Thm (Tukia) $G \subseteq QC(S^n)$ uniform & acts c.c on dist. triples

then $\exists\ g \in QC(S^n)$ st.

$$g G g^{-1} \subseteq Conf(S^n).$$

i.e $\exists g$ st $g\varphi(\Lambda)g^{-1} \subseteq Isom(\mathbb{H}^{n+1})$

so have action of Λ by isoms

Next lecture • Generalize Tukia to boundaries of certain NCHS.

• QI rigidity of lattices in solvable Lie gp

e.g $\mathbb{R} \rtimes \mathbb{R}$ $\quad t \longmapsto e^t$

$$ds^2 = dt^2 + e^{-2t}dx^2$$