Global well-posedness for the Cubic Dirac equation in the critical space.

joint work with S. Herr

Cubic Dirac

For $M > 0$, the cubic Dirac equation for the spinor field $\psi : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{C}^2$ is given by

$$(-i\gamma^\mu \partial_\mu + M)\psi = \langle \gamma^0 \psi, \psi \rangle \psi.$$

$\gamma^\mu \in \mathbb{C}^{2\times 2}$ are the Dirac matrices given by

$$\beta = \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \gamma^2 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}.$$

The $\langle \cdot, \cdot \rangle$ is the standard scalar product on $\mathbb{C}^2$. The cubic Dirac equation can be written for all dimensions by adapting the set of Dirac matrices.

The equation was proposed by Soler as a toy model for self-interacting electron. More fundamental, it is a natural simplification of the Dirac-Maxwell system.

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There are two ways to read the equation in a more familiar fashion.

Let $\mathcal{D}_M = (-i\gamma^\mu \partial_\mu + M)$. Then $\mathcal{D}_M^* = i\gamma^0\partial_t - i\gamma^i\partial_i + M$ and $\bar{\mathcal{D}}_M = \mathcal{D}_M^*\gamma_0$ satisfies

$$\bar{\mathcal{D}}_M \mathcal{D}_M = \gamma^0(\Box + M^2)$$

If we write $\psi = \bar{\mathcal{D}}_M w$ (Klainerman-Machedon), the equation becomes

$$(\Box + M^2)w = Q(Dw, Dw, Dw) + l.o.t.$$

which is a Klein-Gordon equation with a derivative nonlinearity.

Alternatively one applies a projector type operator to the equation to obtain a cubic half-Klein-Gordon system (D'Ancona et all) :

$$(i\partial_t \pm \langle D \rangle)\psi_\pm = "\psi_\pm^3"$$

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The problem in n dimensions is critical in $H^{\frac{n-1}{2}}$.

$n = 3$: Escobado-Vega proved LWP for small data in $H^{1+\epsilon}$. Machihara-Nakanishi-Ozawa proved GWP for small data in $H^{1+\epsilon}$, Machihara-Nakamura-Nakanishi-Ozawa proved GWP for small radial data H^1. B.-Herr proved proved GWP for small data H^1.

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$M = 0$: Bournaveas and Candy proved GWP for small data in the critical space in dimension $n = 2, 3$. They obtain LWP for $M \neq 0$ in the critical space.

Theorem

(B., Herr) The cubic Dirac equation in 2D with $M \neq 0$ is globally well-posed and scatters for small initial data in $H^{\frac{1}{2}}(\mathbb{R}^2)$.

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What is the difference between $M = 0$ and $M \neq 0$?

In toy model the massless cubic Dirac has the form :

$$\Box w = Dw \cdot Dw \cdot Dw$$

which is similar to the Wave Maps equation in toy model

$$\Box \phi = \phi (\nabla \phi)^2 .$$

Bournaveas and Candy approach : the spaces introduced by Tataru work and the better derivative distribution does not require renormalization. A high modulation structure, introduced by B. - Herr in the $3D$ problem, solves the summation problem.

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The basic idea is using the toy model the half-Klein-Gordon with cubic nonlinearity :

$$(i\partial_t \pm \langle D \rangle)\psi_\pm = "\psi_\pm^3"$$

and run an iteration scheme based on the estimate :

$$L_t^2 L_x^\infty \cdot L_t^2 L_x^\infty \cdot L_t^\infty L_x^2 \to L_t^1 L_x^2. \tag{1}$$

There is not much room to modify the scheme since the use of any Strichartz estimate, other than the energy estimate $L^\infty L^2$, would lose derivatives which is a problem in high frequency : this is a half-wave equation, no derivative is recovered when solving the inhomogeneous equation.

Bottom line : an estimate of type (1) should be part of the picture.

In fact, the focus should be on the bilinear L^2 estimate :

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The $L_t^2 L_x^\infty$ estimate.

In high frequency limit, the fundamental solution of the half-Klein-Gordon exhibits decay of type $t^{-\frac{n-1}{2}}$. Using the TT^* argument, deriving the $L_t^2 L_x^\infty$ estimate amounts to $\langle t \rangle^{-\frac{n-1}{2}} \in L_t^1$, thus we need $n \geq 4$.

In low frequency, the fundamental solution of the half-Klein-Gordon exhibits decay of type $t^{-\frac{n}{2}}$. Using the TT^* argument, deriving the $L_t^2 L_x^\infty$ estimate amounts to $\langle t \rangle^{-\frac{n}{2}} \in L_t^1$, thus we need $n \geq 3$.

Natural question : are these real obstructions ? Answer : Yes. Montgomery-Smith proves that the estimates

$$\| P e^{it|\nabla|} f \|_{L_t^2 L_x^\infty} \lesssim \| f \|_{L^2(\mathbb{R}^3)}, \qquad \| P e^{it\Delta} f \|_{L_t^2 L_x^\infty} \lesssim \| f \|_{L^2(\mathbb{R}^2)}$$

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where P, P' project onto transversal frequencies $\lesssim 1$.

Such a setup is known to yield better estimates

$$\|Pe^{it\Delta}f \cdot P'e^{it\Delta}g\|_{L^{\frac{5}{3}}_{t,x}} \lesssim \|f\|_{L^2(\mathbb{R}^2)}\|g\|_{L^2(\mathbb{R}^2)}$$

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Bilinear estimates for free solutions in $2D$.

We need a theory that matches the bilinear estimate for free solutions :

$$\|e^{it\langle D\rangle} f \cdot e^{it\langle D\rangle} g\|_{L^2} \lesssim \|f\|_{L^2}\|g\|_{L^2}.$$

Assume that f, g are supported at frequency $2^{k_1}, 2^{k_2}$ respectively, $k_1 \leq k_2$, and make an angle $\alpha \gg 2^{-k_1}$ between their supports, then

$$\|e^{it\langle D\rangle} f \cdot e^{it\langle D\rangle} g\|_{L^2} \lesssim 2^{\frac{k_1}{2}} \alpha^{-\frac{1}{2}} \|f\|_{L^2}\|g\|_{L^2}.$$

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$$\|e^{it\langle D\rangle} f \cdot e^{it\langle D\rangle} g\|_{L^2} \lesssim 2^{k_1} \|f\|_{L^2}\|g\|_{L^2}.$$

Basic idea : the characteristic surfaces always make an angle. Either $\alpha \lesssim 2^{-k_1}$ or they make an angle of 2^{-2k_1} in the time frequency direction.

Taking into account the null condition which penalizes the interaction by a factor of $\alpha + 2^{-k_1}$ we would get

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We need a theory that matches the bilinear estimate for free solutions :

$$\|e^{it\langle D\rangle} f \cdot e^{it\langle D\rangle} g\|_{L^2} \lesssim \|f\|_{L^2}\|g\|_{L^2}.$$

Assume that f, g are supported at frequency $2^{k_1}, 2^{k_2}$ respectively, $k_1 \leq k_2$, and make an angle $\alpha \gg 2^{-k_1}$ between their supports, then

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Basic idea : the characteristic surfaces always make an angle. Either $\alpha \lesssim 2^{-k_1}$ or they make an angle of 2^{-2k_1} in the time frequency direction.

Taking into account the null condition which penalizes the interaction by a factor of $\alpha + 2^{-k_1}$ we would get

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From the nonlinear equation point of view, the above scheme applies only to the first iteration ! A more robust approach is needed to make all the iterations work.

The goal is to develop a space structure X which contains enough information to capture the above bilinear L^2 estimate :

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where f, g have the appropriate frequency localization.

Natural candidates for X are Strichartz estimates. The problem comes from that using Strichartz estimates other than energy type estimates $L^\infty L^2$ for g (high frequency) would produce powers of 2^{k_2} and this is not acceptable ! Using $L^\infty L^2$ estimates for g requires the use of $L^2 L^\infty$ estimates for f.

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Let $n = 2, 3$. One seeks a decomposition :

$$P_k e^{it\langle D\rangle} f = \sum_\alpha e^{it\langle D\rangle} f_\alpha$$

and a system of frames (t_α, x_α) such that

$$\sum_\alpha \| e^{it\langle D\rangle} f_\alpha \|_{L^2_{t_\alpha} L^\infty_{x_\alpha}} \lesssim 2^{\frac{(n-1)k}{2}} \| f \|_{L^2}.$$

Need a lot of flexibility in energy estimates :

$$\| e^{it\langle D\rangle} Pg \|_{L^\infty_{t_\alpha} L^2_{x_\alpha}} \lesssim C \| Pg \|_{L^2}$$

Here C reflects the angular separation of the support of $\hat{f}_\alpha$ and $\hat{Pg}$. The scheme is closed as follows

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Classical approach for Strichartz estimates.

For solutions localized at frequency 2^k, seek an estimate of type

$$\|e^{it\langle D\rangle}u_0\|_{L^2_t L^\infty_x} \lesssim C(k)\|u_0\|_{L^2}$$

The scaling (in high frequency) indicates that $C(k) = 2^{\frac{(n-1)k}{2}}$.
By TT^* argument, this is follows from an estimate of type

$$\|K_k(t,x)\|_{L^1_t L^\infty_x} \lesssim C^2(k)$$

where

$$K_k(t,x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{it\langle\xi\rangle} \chi_k^2(|\xi|)\, d\xi.$$

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One needs decay type estimates on K_k which are obtained by using standard oscillatory type arguments. The principal curvatures of the characteristic surface $\tau = \sqrt{\xi^2+1}$ play a crucial role.

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Kernel estimates and consequences

The characteristic surface is $\tau = \sqrt{\xi^2 + 1}$ is parabola like for $|\xi| \leq 1$:

$$|K_{\leq 0}(t,x)| \lesssim (1+|t|)^{-\frac{n}{2}}.$$

Low frequencies exhibit Schrödinger type decay. The L^2L^∞ type estimate is dictated by the Schrödinger equation and this is well-understood.

In high frequency the characteristic surface is cone-like, yet it has nonvanishing principal curvatures : two are ≈ 1, the third one is $\approx 2^{-2k}$ (after rescaling). The following bound holds true

$$|K_k(t,x)| \lesssim 2^{nk}(1+2^k|t|)^{-\frac{n-1}{2}} \min(1,(1+2^k|t|)^{-\frac{1}{2}}2^k))$$

There are two decay regimes :
1) $|t| \leq 2^k$ the decay is $t^{-\frac{n-1}{2}}$ (Wave),
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If $n = 3$, the above estimate gives

$$\|K_k\|_{L^1_t L^\infty_x} \lesssim k2^k, \qquad k \gg 1.$$

This gives the end-point Strichartz estimate with logarithmic loss :

$$\|e^{it\langle D\rangle} u_0\|_{L^2_t L^\infty_x} \lesssim k^{\frac{1}{2}} 2^k \|u_0\|_{L^2}, \qquad k \gg 1.$$

which is suboptimal, but good enough to close subcritical ranges.

If $n = 2$, even in the better Schrödinger regime, the decay is t^{-1} hence no estimate of type $\|K_k\|_{L^1_t L^\infty_x}$ is available.

The aim of our works was to come up with an effective replacement for the missing $L^2_t L^\infty_x$ estimate.

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Refined analysis of the kernel.

We recover an L^2L^∞ estimates in adapted systems of coordinates. This is done by localizing the kernel K_k more and understanding its decay properties relative to this localization.

We denote by $\mathcal{K}_l$ a collection of spherical caps of diameter 2^{-l} which "nicely" cover of the unit sphere $\mathbb{S}^{n-1}$. For $\kappa \in \mathcal{K}_l$ let $\omega(\kappa)$ be the center of κ and η_κ be a smooth approximation of the characteristic function of κ.

Fix $k > 0$. For $\kappa \in \mathcal{K}_l$ let

$$K_{k,\kappa}(t,x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{it\langle\xi\rangle} \chi_k^2(|\xi|)\eta_\kappa^2(\xi)\, d\xi.$$

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With $\lambda(k) = (1 + 2^{-2k})^{-\frac{1}{2}}$ and $\omega(\kappa)$ construct the new coordinates (t_Θ, x_Θ). The following estimates hold true for $n = 3$:

$$|K_{k,\kappa}(t,x)| \lesssim 2^k (1 + 2^{-k}|(t,x)|)^{-\frac{3}{2}}.$$

$$|K_{k,\kappa}(t,x)| \lesssim_N 2^k (1 + 2^k |t_\Theta|)^{-N}, \qquad |t_\Theta| \gg 2^{-2k}|(t,x)|.$$

As a consequence we obtain

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Adapted coordinates. Given an angle ω and a parameter λ, define the directions $\Theta_{\lambda,\omega} = \frac{1}{\sqrt{1+\lambda^2}}(\lambda,\omega)$, $\Theta^{\perp}_{\lambda,\omega} = \frac{1}{\sqrt{1+\lambda^2}}(-1,\lambda\omega)$ and the associated orthogonal coordinates $(t_\Theta, x^1_\Theta, x'_\Theta)$

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$$2^{-k}\|e^{it\langle D\rangle}u_0\|_{L^2_tL^\infty_x} + \|e^{it\langle D\rangle}u_0\|_{L^2_{t_\Theta}L^\infty_{x_\Theta}} \lesssim \|u_0\|_{L^2}.$$

for u_0 localized at frequency 2^k and cap κ.

The Strichartz estimate in adapted frames adds in a favorable way with respect to caps

$$\sum_{\kappa\in\mathcal{K}_k}\|e^{it\langle D\rangle}P_\kappa u_0\|_{L^2_{t_{\Theta_\kappa}}L^\infty_{x_{\Theta_k}}} \lesssim 2^k\|u_0\|_{L^2}.$$

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If $n = 2$ the kernel estimate becomes :

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The problem now is that in the regime $|t| \geq 2^k$ the decay is too weak.

The fix comes by exploiting the decay of $|t|^{-1}$ in a different fashion inspired by the work on the $2D$ Schrödinger equation.

For $T \leq 2^r, r \in \mathbb{N}$, for $k \geq 100$, and $\kappa \in \mathcal{K}_k$ we define

$$\Lambda_{k,\kappa} = \left\{ \frac{1}{\sqrt{1 + m^{-2}}}; m \in 2^{-r-10}\mathbb{Z} \cap [2^{k-3}, 2^{k+3}] \right\} \times \{\omega(\kappa)\}$$

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we obtain that for $f \in L^2(\mathbb{R}^2)$ supported at frequency 2^k in the the cap κ,

$$\|1_{[-T,T]}(t) e^{it\langle D \rangle} f\|_{\sum_{\Lambda_{k,\kappa}} L^2_{t_\Theta} L^\infty_{x_\Theta}} \lesssim \|f\|_{L^2},$$

and this adds correctly to give the factor predicted by scaling

$$\sum_{\kappa \in \mathcal{K}_k} \|1_{[-T,T]}(t) e^{it\langle D \rangle} \tilde{P}_\kappa f\|_{\sum_{\Lambda_{k,\kappa}} L^2_{t_\Theta} L^\infty_{x_\Theta}} \lesssim 2^{\frac{k}{2}} \|f\|_{L^2}.$$

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$$\Omega_{k,\kappa} = \{\lambda(k)\} \times \left\{ R^i\omega(\kappa); i \in \mathbb{Z}, |i| \le 2^{-k-8+r} \right\}$$

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Energy estimates

The Strichartz estimates need to be paired with corresponding energy estimates, that is $L^\infty_{t_\Theta} L^2_{x_\Theta}$. Given two sets of parameters (k_1, κ_1) and (k_2, κ_2) with $k_1 \leq k_2$ and (k_1, κ_1) generating the direction Θ, one needs an energy estimate of type

$$\|e^{it\langle D\rangle} u_0\|_{L^\infty_{t_\Theta} L^2_{x_\Theta}} \lesssim C(k_1, k_2, \kappa_1, \kappa_2) \|u_0\|_{L^2}.$$

This is doable provided that : $\alpha = d(\kappa_1, \kappa_2) \gg 2^{-k_1}$ in which case

$$C = \alpha^{-1},$$

as well as when $d(\kappa_1, \kappa_2) \ll 2^{-k_1}$ in which case

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One uses instead the other set of coordinates

$$\|e^{it\langle D\rangle}u_0\|_{L^\infty_{x^2_\Theta}L^2_{(t,x^1)_\Theta}} \lesssim 2^{\frac{k_1}{2}}\|u_0\|_{L^2}.$$

Toy model for closing the argument. Via a duality argument, one needs to estimate

$$\int \langle\psi,\beta\psi\rangle\cdot\langle\psi,\beta\psi\rangle dxdt.$$

It is enough to estimate

$$\|\langle\psi,\beta\psi\rangle\|_{L^2}.$$

Though not apparent, there is a null structure in this bilinear form which is of the order of the angular separation of the interacting frequencies.

The estimate is morally of the form :

$$L^2_{t_\Theta}L^\infty_{x_\Theta}\cdot L^\infty_{t_\Theta}L^2_{x_\Theta}\to L^2.$$

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