Motion of a Random String:

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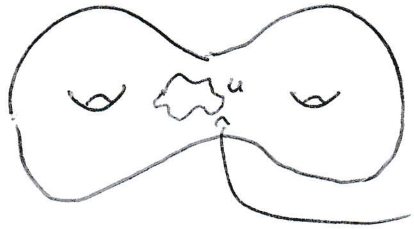
Joint with L. Zambotti, Y. Bruned (work in progress)

n-dim smooth manifold M



analogue of heat eqn
converts curve to a geodesic

$$u : S^1 \longrightarrow M \quad , \qquad E(u) = \int g_{u(t)}(\dot{u}(t), \dot{u}(t))\, dt$$

Minimizers are geodesics

$\Longrightarrow$ consider gradient flow

Gradient flow of E: $\quad \partial_t u^\alpha = \partial_x^2 u^\alpha + \Gamma^\alpha_{\beta\eta}(u)\, \partial_x u^\beta \partial_x u^\eta$

<u>Q</u>: Random analogue of this?

<u>Wanted</u>: Evolution with "$\exp(-E(u))\, du$" as invariant measure.

interpretation is subtle

<u>In $\mathbb{R}^n$</u>: $\quad \partial_t u = \partial_x^2 u + \xi$ space-time white noise, $\xi \in C^{-\frac{3}{2}-}$

$$\mathbb{E}(\xi(x,t)\xi(y,s)) = \delta(x-y)\,\delta(t-s)$$

Brownian bridge is invariant, forward and backward in time.

<u>On M</u>: $\quad \partial_t u^\alpha = \partial_x^2 u^\alpha + \Gamma^\alpha_{\beta\eta}\, \partial_x u^\beta \partial_x u^\eta + \sigma_i^\alpha \xi_i$

with $\sigma_i^\alpha \sigma_i^\beta = g^{\alpha\beta}$, σ_i on $\mathcal{M}$. (unique ?)

It turns out that we will also want to impose

(1). $\sum_i \nabla_{\sigma_i} \sigma_i = 0$, can be done by embedding into a higher-dimensional manifold.

Thm. ξ_i^ε with $\mathbb{E}\left(\xi_i^\varepsilon(x,t)\,\xi_j^\varepsilon(y,s)\right) = \delta_{ij}\,\varepsilon^{-3}\rho\left(\frac{x-y}{\varepsilon}, \frac{t-s}{\varepsilon^2}\right)$

$\int \rho = 1$.

Then $u_\varepsilon \longrightarrow u$ up to a stopping time

u might depend on ρ.

(corresponds to a different choices of an extra term in)

Consider $\boxed{\partial_t u_\varepsilon^\alpha = \partial_x^2 u_\varepsilon^\alpha + \Gamma^\alpha_{\beta\mu}(u_\varepsilon)\,\partial_x u_\varepsilon^\beta\,\partial_x u_\varepsilon^\mu + \sigma_i^\alpha(u)\,\xi_i^\varepsilon}$

NOTE: This type of result is not true in general, based on using Levi-Civita connection.

Arbitrary connection $\Longrightarrow$ extra $-\frac{\log \varepsilon}{4\pi\sqrt{3}}\,\underbrace{R(\nabla_G G)^\alpha}_{=0 \text{ for Levi-Civita}}$

required for convergence

scaling $\longrightarrow$ at small scales, should consider nonlinear part small.

Consider $\quad \partial_t v_i = \partial_x^2 v_i + \xi_i^\varepsilon$

Guess, at small scales solutions should look like

$$u^\alpha(y,s) \simeq u^\alpha(x,t) + \sigma_i^\alpha(u(x,t))\,(v_i(y,s) - v_i(x,t))$$

$$\mathbb{E}(v_i^\varepsilon \xi_i^\varepsilon) \sim -\frac{c}{\varepsilon}, \quad \mathbb{E}\,\partial_x v_i^\varepsilon \partial_x v_i^\varepsilon \sim \frac{c}{\varepsilon}$$

exactly cancel out by (1).

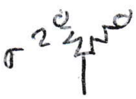
Need a higher order version of this expansion

Notation: $\circ$: instance of noise ξ^ε

— : convolution with P (heat kernel).

$\sim$: convolution with $\partial_x P$

$v(y,s) - v(x,t)$:  $\quad \partial_x v$:  $\quad$  $\longleftarrow (\partial_x v)^2$

Ansatz: $U = u\,\mathbb{1} + \sigma$  $+ 2\partial\sigma$  $+ \Gamma\sigma^2$ 

$+ 2\Gamma\sigma^2\partial\sigma$  $+ \dots$

Individual terms don't have limits as $\varepsilon \to 0$.

Hairer

Step 2: Redefine these objects to force them to converge.

Want $M_\varepsilon : T \longrightarrow T$ ← set of formal expressions of type above.

to force convergence.

E.g. $M_\varepsilon :$  $\longmapsto$  $- \frac{c}{\varepsilon} \mathbb{1}$.

(Resembles a technique from QFT.)

Author initials,
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'68 80's.

Also want M_ε to preserve the combinatorial structure of these objects.

Also Connes Kreimer 99-2000

One considers $U = \sum_\tau' u_\tau \cdot \tau$, u_τ new unknown coefficients.

12 expressions of type above.

A fixed point argument is done in this framework.