Invariant measures for nonlinear PDE

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Plan for the Lectures

In these lectures, I will try to convey the main ideas behind Bourgain's approach to invariant measures –after the works of Lebowitz-Rose-Speer and of Zhidkov – and to almost sure global well-posedness for dispersive PDE.

I will not be able to go over all details or precise definitions but rather intend to give you the main flavors of what's involved.

In these lectures we will focus on **three prototypes**:

- The focusing quintic nonlinear Schrödinger equation (NLS) on $\mathbb{T}$.
- The derivative NLS equation (DNLS) on $\mathbb{T}$.
- The defocusing cubic NLS on $\mathbb{T}^2$.
  - If time permits, will discuss some further questions on Lecture 3.

Much work has been done in recent years by many people for a variety of nonlinear wave and dispersive equations and systems and in different contexts. I will try to point to a few references.

The NLS on $\mathbb{T}^d$

$\mathbb{T}^d := (\mathbb{R}/\mathbb{Z})^d$, the square d-dimensional torus of periodic functions and $p > 1$

$$\text{(p-NLS)} \qquad \begin{cases} iu_t + \Delta u = \pm |u|^{p-1} u, \\ u(0,x) = \phi(x) \in H^s, \qquad x \in \mathbb{T}^d, \end{cases}$$

We are working on the square torus. Many interesting problems on *irrational tori*:

$$\Lambda_d(\beta) := (\mathbb{R}/\beta_1\mathbb{Z}) \times (\mathbb{R}/\beta_2\mathbb{Z}) \times \cdots \times (\mathbb{R}/\beta_d\mathbb{Z})$$

where $\beta := (\beta_1, \beta_2, \ldots, \beta_d)$, $\beta_j > 0$, $j = 1, \ldots d$ and at least one of the ratios $\frac{\beta_j}{\beta_{j'}} \notin \mathbb{Q}$, $j \neq j'$. Irrational tori come up naturally experimentally, as well as in KAM theory and Hamiltonian chaos (Bourgain 2007).

Conserved quantities

Mass:

$$M(u(t)) := \int |u(t,x)|^2 \, dx$$

Hamiltonian:

$$H(u(t)) := \frac{1}{2} \int |\nabla u(t,x)|^2 \, dx \pm \frac{1}{p+1} \int |u(t,x)|^{p+1} dx$$

are both constant in time.

- The $+$ case is called *defocusing*.
  The Hamiltonian and the mass give a global in time bound for the H^1 norm of $u(t,x)$.

- The $-$ is called *focusing*. The energy could be negative and blow up may occur.

Local well posedness (deterministic)

We say that the Cauchy IVP is LWP in H^s if for any ball B in H^s there exists a time $\tau > 0$ and a Banach space $X^s \subset C([-\tau,\tau]; H^s)$, s.t. for each initial data $\phi \in B$ there exists a unique $u \in X^s \cap C([-\tau,\tau]; H^s)$ of the integral equation (Duhamel pple.):

$$u(x,t) = S(t)\phi(x) + c_\lambda \int_0^t S(t-t')\,|u(t',x)|^{p-1}u(t',x)\,dt'.$$

Moreover, the map $\phi \longmapsto u$ is continuous from H^s into $C([-\tau,\tau];\ H^s)$. If $\tau > 0$ can be taken arbitrarily large, then we say that the initial value problem is *globally* well posed.

Note uniqueness is on $X^s \cap C([-\tau,\tau];\ H^s)$. Proving uniqueness on $C([-\tau,\tau];\ H^s)$ requires additional work and when it holds, the local well posedness is said to be *unconditional*.

Small detour: scaling symmetry on $\mathbb{R}^d$

$$\phi(x) \longmapsto \mu^{-\frac{2}{(p-1)}}\phi(\frac{x}{\mu}) =: \phi_\mu(x)$$

$$u(t,x) \longmapsto \mu^{-\frac{2}{(p-1)}}u(\frac{t}{\mu^2},\frac{x}{\mu}), \qquad \mu > 0$$

Then if $\phi \in \dot{H}^{s_c}$ where $s_c := \frac{d}{2} - \frac{2}{p-1}$, we have

$$\|\phi_\mu\|_{\dot{H}^{s_c}} = \|\phi\|_{\dot{H}^{s_c}}$$

and the equation is scale invariant. s_c is called the *critical* scaling regularity.

Relative to scaling, data in H^s is said to be

- *subcritical* if $s > s_c$ (large data local well posedness)
- *critical* if $s = s_c$ (hard/ small data well posedness)
- *supercritical* if $s < s_c$ (scaling against us/ small data might lead to 'bad' behavior in short times)

- p-NLS also enjoys time-reversibility: $\phi(x) \to \overline{\phi(x)}$, $u(t,x) \to \overline{u(-t,x)}$,

- To prove local well posedness for p-NLS one needs to find a suitable Banach space X^s on which to prove that the map

$$\Phi : \mathfrak{u} \longmapsto S(t)\phi + c\int_0^t S(t-t')\,|\mathfrak{u}(t',x)|^{p-1}\mathfrak{u}(t',x)\,dt'$$

is a **contraction**, whence its **fixed point** is the solution $u(t,x)$.

- Determining a good choice of X^s is part of the problem... It is dictated by being able to prove good estimates for $v := S(t)\phi = e^{it\Delta}\phi$ the solution to the **linear evolution**,

$$\begin{cases} iv_t + \Delta v \;=\; 0 \\ v(0,x) = \phi(x). \end{cases}$$

on such space so that then, **at least in the subcritical regime** one can show that $\Phi(\mathfrak{u})$ - hence the solution u- satisfy similar estimates (locally in time).

More precisely, to prove the contraction we then need to show:

$$\|\Phi(\mathfrak{u})\|_{X^s} \leq \; c\|\phi\|_{H^s} + \tau^{\alpha}\|\mathfrak{u}\|_{X^s}^{p}$$

$$\|\Phi(\mathfrak{u}) - \Phi(\mathfrak{v})\|_{X^s} \leq \tau^{\alpha}(\|\mathfrak{u}\|_{X^s} + \|\mathfrak{v}\|_{X^s})^{p-1}\|\mathfrak{u} - \mathfrak{v}\|_{X^s}$$

Then on the ball in X^s of radius -say- $R \sim 2c\|\phi\|_{H^s}$ we have a contraction provided we choose $\tau > 0$ small enough, i.e.

$$\tau \quad \sim \quad \frac{c_{s,p}}{\|\phi\|_{H^s}^{C_{s,p}}},$$

Strichartz estimates on tori

The most basic and important space-time estimates that $v = S(t)\phi$ the solution to the **linear problem** satisfy are the so called *Strichartz estimates*:

Theorem (Bourgain-Demeter)

For $N \geq 1$, let $\phi \in L^2(\mathbb{T}^d)$ be a smooth function such that the $\operatorname{supp} \hat{\phi} \subset [-N, N]^d \subset \mathbb{Z}^d$. *Then for any $\epsilon > 0$ the following estimates hold:*

$$\|S(t)\phi\|_{L^q_t L^q_x(\mathbb{T}^{d+1})} \lesssim C_q \|\phi\|_{L^2_x(\mathbb{T}^d)} \quad \text{if} \quad q < \frac{2(d+2)}{d}$$

$$\|S(t)\phi\|_{L^q_t L^q_x(\mathbb{T}^{d+1})} \ll N^{\epsilon} \|\phi\|_{L^2_x(\mathbb{T}^d)} \quad \text{if} \quad q = \frac{2(d+2)}{d}$$

$$\|S(t)\phi\|_{L^q_t L^q_x(\mathbb{T}^{d+1})} \lesssim C_q N^{\frac{d}{2} - \frac{d+2}{q}} \|\phi\|_{L^2_x(\mathbb{T}^d)} \quad \text{if} \quad q > \frac{2(d+2)}{d}$$

In the periodic setting, proving these was highly nontrivial and required new ideas than those used on $\mathbb{R}^d$. They were introduced by Bourgain for the rational torus as a conjecture.

Bourgain proved some and then recently **Bourgain-Demeter (14')** obtained the full range for rational and irrational tori as well (see also **Killip-Visan**).

- $d = 1$, $q < 6$; we have for ex. $\|S(t)\phi\|_{L^4_{xt}(\mathbb{T}\times\mathbb{T})} \lesssim C\|\phi\|_{L^2_x(\mathbb{T})}$

- $d = 1$, $q = 6$ gives $\|S(t)\phi\|_{L^6_{xt}(\mathbb{T}\times\mathbb{T})} \ll N^\epsilon\|\phi\|_{L^2_x(\mathbb{T})}$

- $d = 2$, $q = 4$ gives $\|S(t)\phi\|_{L^4_{xt}(\mathbb{T}^2\times\mathbb{T})} \ll N^\epsilon\|\phi\|_{L^2_x(\mathbb{T}^2)}$

- Bourgain showed that dispersion is indeed weaker in the periodic setting. He proved that in $1D$, the L^6 estimate with constant independent of N is **false** ($(\log N)^{1/6}$ growth).

- These ϵ-loss will prevent us to close the estimates in $L^2(\mathbb{T}^d)$ for the quintic NLS in 1D and the cubic in 2D. Need $s > 0$.

Well-posedness for p-NLS on $\mathbb{T}$ and $\mathbb{T}^2$ (Bourgain 93')

Theorem ($d = 1, p = 3$)

The cubic NLS on $\mathbb{T}$ (either focusing or defocusing) is GWP in $L^2(\mathbb{T})$. Also in $H^s(\mathbb{T})$, $s \geq 0$ (preservation of regularity).

Theorem ($d = 1, p = 5$)

The quintic NLS on $\mathbb{T}$ (either focusing or defocusing) is LWP in $H^s(\mathbb{T})$, $s > 0$ (time of existence depends on $\|\phi\|_{H^s}^{-C}$).

Theorem ($d = 2, p = 3$)

The <u>defocusing</u> cubic NLS on $\mathbb{T}^2$ is LWP in $H^s(\mathbb{T}^2)$, $s > 0$.

In the last two theorems and for the defocusing case one gets GWP for $s \geq 1$.

Invariant Measures and almost sure GWP.

J. Bourgain's (90's) studied dynamics of periodic dispersive equations (NLS, KdV, mKdV, Zakharov system) through

- the introduction and use of the Gibbs measure derived from the PDE viewed as an infinite dimensional Hamiltonian system.

Global in time existence was studied in the almost sure sense in rather low regularity regimes via the *existence and invariance* of the associated Gibbs measure (cf. Lebowitz, Rose and Speer's and Zhidkov's works).

Let μ be a probability measure on the space of initial data $\mathcal{H}^s$.

Informal Definition- Almost sure global well-posedness

There exists $\Sigma \subset \mathcal{H}^s$, with $\mu(\Sigma) = 1$ and such that for any $u_0 \in \Sigma$ the IVP with data u_0 is globally wellposed.

In other words, well posedness in the support of the measure, supp $\mu \subset \mathcal{H}^s$ and hence almost surely in $\mathcal{H}^s$.

- Why invariant Gibbs measures are effective in gwp?
  - The **invariance** of the Gibbs measure, just like the usual conserved quantities, can be used to **control the growth in time** of those solutions in its support and **extend the local in time solutions to global ones** almost surely.
  - In some cases, failure to show global existence might come from certain 'exceptional' initial data set, and the virtue of the Gibbs measure lies in that it would not see 'exceptional sets' of initial data (capturing 'generic behaviour'.)
  - Invariant measures for infinite dimensional Hamiltonian system are also important in their own right. Given an invariant measure, we can view the system on phase space as a dynamical system with a measure preserving transformation Φ=solution map $u_0(\cdot) \longrightarrow u(1,\cdot)$ for which the **Poincaré recurrence theorem** follows. A very interesting but quite hard question is whether one can prove the **ergodicity** of the (nonlinear) system?

- What are the limitations of this approach to gwp ?
  - The difficulty lies in the actual construction of the associated Gibbs or weighted Wiener measures and in showing both, its invariance under the flow and the almost sure global well-posedness.
  - Measures are easier to construct on bounded domains ($\mathbb{T}^d$, the ball, compact Riemannian manifolds, etc.)
    - For existence of invariant measures on $\mathbb{R}$ see Bourgain 00' (NLS); McKean-Vaninsky 95' (NLW). Also Burq-Thomann-Tzvetkov 10' and Y. Deng (11') (NLS with potential 1D and 2D radial resp.), S. Xu 14' (3D radial NLW).
  - **Not available in higher dimensions (no symmetry assumptions on manifold).**
    The standard construction is not normalizable and the support would live in very rough spaces (estimates are not available).
  - If the PDE is not Hamiltonian? Not clear....

After Bourgain's work in 94-96', the approach to a.s GWP via invariant measures was re-taken again in 07-08' (Tzvetkov, Burq-Tzvetkov, T. Oh). Lots of activity:

- Schrödinger Equations: Bourgain, Tzvetkov, Thomann, Thomann-Tzvetkov, A.N.-Oh-Rey-Bellet-Staffilani, A.N.-Rey-Bellet-Sheffield-Staffilani, Burq-Thomann-Tzevtkov, Y. Deng, Burq-Lebeau, Bourgain-Bulut,
- NLW and NLKG Equations: Burq-Tzvetkov, de Suzzoni, Bourgain-Bulut, S. Xu
- gKdV Equations: Bourgain, Oh-Quastel-Valko, T. Oh, Richards (cf. also Quastel-Valko, Cambronero-McKean)
- Benjamin-Ono Equations: Y. Deng, Tzvetkov-Visciglia. and Y. Deng-Tzvetkov-Visciglia.
- Others . . .

The 5-NLS on $\mathbb{T}$: invariant Gibbs measure and a.s gwp

(NLS) $$\begin{cases} iu_t + \Delta u = \pm|u|^4 u, & \\ u(0,x) = \phi(x) \in H^s, & x \in \mathbb{T}, \end{cases}$$

We will be particularly interested in the focusing case when the construction of the measure as a weighted Wiener measure is more delicate ([LRS, B]).

Before getting down to the details we review some basic facts.

Gaussian measures in Hilbert spaces

Let H, *real* separable Hilbert space

$B : H \to H$, linear, positive, self-adjoint operator

$\{e_n\}_{n=1}^{\infty}$, eigenvectors of B forming an O.N. basis of H

$\{\lambda_n\}_{n=1}^{\infty}$, corresponding eigenvalues

Define

$$\rho_N(M) = (2\pi)^{-\frac{N}{2}} \Big(\prod_{n=1}^{N} \lambda_n^{-\frac{1}{2}} \Big) \int_F e^{-\frac{1}{2}\sum_{n=1}^{N} \lambda_n^{-1} x_n^2} \prod_{n=1}^{N} dx_n$$

where M is a cylindrical set in H: i.e. there exists N and a Borel set F in $\mathbb{R}^N$ s.t. $\begin{cases} M = \{x \in H : (x_1, \cdots, x_N) \in F\}, \\ x_n := \langle x, e_n \rangle_H = \text{n-th coordinate of } x. \end{cases}$

i.e. ρ_N is a Gaussian measure on $E_N = \text{span}\{e_1, \cdots, e_N\}$.

Note $\mathcal{A} := \{M : M \subset H \text{ is cylindrical } \}$ then $\mathcal{A}$ is a field.

Define ρ on H by $\rho\big|_{E_N} = \rho_N$.

Facts

(1) ρ is countably additive on $\mathcal{A}$ if and only if B is of trace class, i.e.

$$\sum \lambda_n < \infty.$$

(2) $\rho_N \rightharpoonup \rho$ as $N \to \infty$.

Also if (1) holds, the minimal σ-field $\mathcal{M}$ containing $\mathcal{A}$ is the Borel σ-field on H.

Remark

We will see that in fact in many instances in PDE we need to realize this measure as a measure supported on a suitable Banach space. This needs some extra work but it is possible by relying on L. Gross 65' and H. Kuo 75' theory of abstract Wiener spaces. More later!

A concrete example on $H^s(\mathbb{T}^d)$

Consider

$$d\rho_N = Z_N^{-1} \exp\Big(-\frac{1}{2} \sum_{|n|\leq N} (1+|n|^2)|\widehat{\phi}_n|^2 \Big) \prod_{|n|\leq N} da_n db_n$$

where $\widehat{\phi}_n = a_n + ib_n$.
We may view ρ_N as an induced probability measure on $\mathbb{C}^{2N+1} \equiv \mathbb{R}^{4N+2}$ under the map

$$\omega \longmapsto \Big\{ \frac{g_n(\omega)}{\sqrt{1+|n|^2}} \Big\}_{|n|\leq N}.$$

where $\{g_n(\omega)\}_{|n|\leq N}$ are i.i.d complex Gaussian random variables (centered) on a probability space $(\Omega, \mathcal{F}, P)$.

Question: As $N \to \infty$ where does ρ, the (weak) limit of ρ_N make sense as a countably additive probability measure?

Consider the operator $B_s := (1-\Delta)^{s-1}$ on $\mathbb{T}^d$ then

$$\sum_{n\in\mathbb{Z}^d}(1+|n|^2)|\widehat{\phi}_n|^2 = \langle \phi, \phi\rangle_{H^1} = \langle B_s^{-1}\phi, \phi\rangle_{H^s}.$$

The operator $B_s : H^s \to H^s$ has eigenvalues $\{(1+|n|^2)^{(s-1)}\}_{n\in\mathbb{Z}^d}$.

Their eigenvectors $\{(1+|n|^2)^{-s/2}e^{in\cdot x}\}_{n\in\mathbb{Z}^d}$ form an o.n basis of $H^s(\mathbb{T}^d)$.

- For ρ to be *countably additive* we need B_s to be of *trace class* which is true if and only if

$$s < 1 - \frac{d}{2}$$

.

- In 1D, ρ is a countably additive measure on $H^s(\mathbb{T})$ for any $s < 1/2$, but **not** for $s \geq 1/2$.
- In 2D, ρ is a countably additive measure on $H^s(\mathbb{T}^2)$ for any $s < 0$, but **not** for $s \geq 0$.
- ......

- We also view ρ as the induced probability measure under the map

$$\omega \mapsto \left\{ \frac{g_n(\omega)}{\sqrt{1+|n|^2}} \right\}_{n \in \mathbb{Z}^d}.$$

- ρ yields for ϕ the distribution of a random (Fourier) series

$$\phi = \phi^\omega = \sum_{n \in \mathbb{Z}^d} \frac{g_n}{\sqrt{1+|n|^2}} e^{in \cdot x}.$$

which defines almost surely a function (if $d = 1$) or a distribution (when $d \geq 2$) in $H^s(\mathbb{T}^d)$, $s < 1 - \frac{d}{2}$. $\quad (\widehat{\phi}_n = \frac{g_n}{\sqrt{1+|n|^2}}$ on the support of ρ.)

Lemma [Fernique-type tail estimate]

There exists $C > 0$, s.t. for all K suff. large, and $s < 1 - \frac{d}{2}$ we have

$$\rho\big(\{\|\phi\|_{H^s} > K\}\big) < e^{-cK^2}.$$

The Gibbs measure: finite dimension

Hamilton's equations of motion have the antisymmetric form

$$\text{(HE)} \qquad \dot{q}_i = \frac{\partial H(p,q)}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H(p,q)}{\partial q_i}$$

the Hamiltonian $H(p,q)$ being a first integral:

$$\frac{dH}{dt} := \sum_i \frac{\partial H}{\partial q_i}\dot{q}_i + \frac{\partial H}{\partial p_i}\dot{p}_i = \sum_i \frac{\partial H}{\partial q_i}\frac{\partial H}{\partial p_i} + \frac{\partial H}{\partial p_i}(-\frac{\partial H}{\partial q_i}) = 0$$

And by defining $y := (q_1, \dots, q_k, p_1, \dots, p_k)^T \in \mathbb{R}^{2k}$ $(2k = d)$ we can rewrite

$$\frac{dy}{dt} = J \nabla H(y), \qquad J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

Liouville's Theorem: Let a vector field $f : \mathbb{R}^d \to \mathbb{R}^d$ be divergence free then the flow map Φ_t of a system is a volume preserving map (for all t).
The flow map satisfies:

$$\frac{d}{dt}\Phi_t(y) \;=\; f(\Phi_t(y)).$$

In particular if f is associated to a Hamiltonian system then automatically $\operatorname{div} f = 0$. Indeed

$$\operatorname{div} f = \frac{\partial}{\partial q_1}\frac{\partial H}{\partial p_1} + \frac{\partial}{\partial q_2}\frac{\partial H}{\partial p_2} + \dots \frac{\partial}{\partial q_k}\frac{\partial H}{\partial p_k} - \frac{\partial}{\partial p_1}\frac{\partial H}{\partial q_1} - \frac{\partial}{\partial p_2}\frac{\partial H}{\partial q_2} - \dots \frac{\partial}{\partial p_k}\frac{\partial H}{\partial q_k} = 0$$

by equality of mixed partial derivatives.
The Lebesgue measure on $\mathbb{R}^{2k}$ is invariant under the Hamiltonian flow (HE).
Consequently from conservation of Hamiltonian H the Gibbs measures,

$$d\mu := e^{-\beta H(p,q)} \prod_{i=1}^{d} dp_i \, dq_i$$

with $\beta > 0$ are invariant under the flow of (HE); ie. for $A \subset \mathbb{R}^d$,

$$\mu(\Phi_t(p_0, q_0) \in A) = \mu((p_0, q_0) \in A)$$

Infinite Dimension Hamiltonian PDEs

Consider the focusing p-NLS on $\mathbb{T}$ for $p \leq 5$

$$iu_t + u_{xx} + |u|^{p-1}u = 0$$

with Hamiltonian

$$H(u) := \frac{1}{2}\int |u_x|^2 - \frac{1}{p+1}\int |u|^{p+1} = \frac{1}{2}\int |u_x|^2 - \mathcal{N}(u).$$

• Think of $u(t)$ as the infinite dimension vector given by its Fourier coefficients $(a_n(t) + ib_n(t))_{n\in\mathbb{Z}}$. The equation above becomes an infinite dimension Hamiltonian system for the vector $(a_n(t), b_n(t))_{n\in\mathbb{Z}}$.

• Lebowitz, Rose and Speer (1988) considered the Gibbs measure *formally* given by

$$`d\mu = Z^{-1}\exp\left(-\beta H(u)\right)\prod_{x\in\mathbb{T}} du(x)'$$

and showed that μ is a well-defined probability measure on $H^s(\mathbb{T}), s < \frac{1}{2}$ but not for $s = \frac{1}{2}$.

- The definition of μ above - although suggestive – is a purely formal expression:
  - It is impossible to define the Lebesgue measure as a countably additive measure on an infinite-dimensional space.
  - Moreover as it'll turn out, $\int |u_x|^2 = \infty$, μ-almost surely.
  - In fact all three factors are infinite.
- The result only holds with an L^2-cutoff $\chi_{\|u\|_{L^2} \leq B}$ (needed to normalize the measure) where:
  - If $p < 5$ any $B > 0$ works.
  - If $p = 5$ **need suff. small** $B > 0$.
- Recall the L^2 norm is conserved.

Invariance of μ

- Bourgain (96') showed the almost sure global well-posedness and the **invariance** of the Gibbs measure μ

  If $\Phi(t)$ denotes the flow map associated to our nonlinear Hamiltonian PDE. Suppose $\Phi(t)$ is defined for all $t \in \mathbb{R}$, μ almost surely.

- We say μ is **invariant** if for all $f \in L^1(\mathcal{H}, \mu)$ and all $t \in \mathbb{R}$,

$$\int f(\Phi(t)(\phi))\, \mu(d\phi) \;=\; \int f(\phi)\mu(d\phi).$$

  In short,

$$\mu(\Phi(-t)(A)) = \mu(A) \qquad\qquad \text{for all } t$$

  and all measurable sets A in $\mathcal{H}$.

How is μ defined? Idea.

One uses a Gaussian measure as reference measure.

Then the weighted measure μ is constructed in two steps:

- First one constructs a Gaussian measure ρ as the limit of the finite-dimensional measures on $\mathbb{R}^{4N+2}$ given by

$$d\rho_N = Z_N^{-1} \exp\Big(-\frac{\beta}{2} \sum_{|n| \leq N} (1 + |n|^2) |\widehat{u}_n|^2 \Big) \prod_{|n| \leq N} da_n db_n$$

  where $\widehat{u}_n = a_n + ib_n$.

- Once ρ has been constructed one constructs μ as a measure which is **absolutely continuous** with respect to ρ and whose Radon-Nikodym derivative is

$$\frac{d\mu}{d\rho} = R(v) := \tilde{Z}^{-1}\chi_{\{\|v\|_L^2 \leq B\}} e^{\frac{\beta}{2}\mathcal{N}(v)}$$

where $\mathcal{N}(v)$ is the potential energy.

For this measure to be normalizable one needs the L^2 cutoff.

For different constants $\beta > 0$, the measures will all be invariant and are all mutually singular.

- $\frac{1}{\beta}$ = temperature in statistical physics.
- We fix $\beta = 1$ (no role).

The details for focusing 5-NLS on $\mathbb{T}$

Let us denote by $\mathcal{H} = H^s(\mathbb{T}), s < \frac{1}{2}$ fixed.

Let P_N be the Fourier/Dirichlet projection onto the spatial frequencies $\leq N$.

Consider the finite dimensional approximation to NLS: :

$$\text{(FDA)} \qquad \begin{cases} iu_t^N + \Delta u^N + P_N(|u^N|^4 u^N) = 0 \\ u^N(0,x) = \phi^N(x) := P_N\phi(x) = \sum_{|n|\leq N} \widehat{\phi}(n)e^{inx}, & x \in \mathbb{T}. \end{cases}$$

Then, for by the deterministic local theory we have that for initial data $\|\phi\|_{\mathcal{H}} \leq K$, the (FDA) is LWP on $[-\tau, \tau]$ with $\tau \sim K^{-C}$, **independent of** N.

Crucial Fact:

- FDA is still a **Hamiltonian system:** $iu_t^N = \frac{dH(u^N)}{d\overline{u^N}}$ with Hamiltonian

$$H(u^N)(t) = \frac{1}{2}\int |u_x^N|^2 dx - \frac{1}{6}\int |u^N|^6 dx$$

  which is still **conserved** under the FDA flow.

- By Liouville's theorem the Lebesgue measure

$$\prod_{|n|\leq N} da_n db_n,$$

where $\widehat{u^N}(n) = a_n + ib_n$, is invariant under the flow of (FDA).

- Then, using the conservation of $H(u^N)$- we have that the finite dimensional version of μ:

$$d\mu_N = \tilde{Z}_N^{-1} e^{-H(u^N)} \prod_{|n|\leq N} da_n db_n$$

is an invariant measure under the flow of (FDA)

- One still needs to prove that μ_N converges weakly to μ and that μ is an invariant measure on $\mathcal{H}$.

- **However** it is the invariance of μ_N what allow us to extend local in time solutions to global ones.

Extending u^N globally in time

Main Proposition [Growth of solutions to FDA] Bourgain '94

Given $T < \infty$, $\varepsilon > 0$, there exists $\Omega_N \subset \mathcal{H}$ s.t.

- $\mu_N((\Omega_N)^c) < \varepsilon$
- for $\phi^N \in \Omega_N$, (FDA) is well-posed on $[-T, T]$ with the growth estimate:

$$\|u^N(t)\|_{\mathcal{H}} \lesssim \left(\log \frac{T}{\varepsilon}\right)^{\frac{1}{2}}, \text{ for } |t| \leq T.$$

In other words, we have that provided the data is in Ω_N the solutions to the FDA extend globally in time.

Proof.

Let $\Phi_N(t)$ = flow map of (FDA), and define the 'good set' of data:

$$\Omega_N = \cap_{j=-[T/\tau]}^{[T/\tau]} \Phi_N(j\tau)(\{\|\phi^N\|_{\mathcal{H}} \leq K\}).$$

- By invariance of μ_N,

  $$\mu(\Omega_N^c) = \sum_{j=-[T/\tau]}^{[T/\tau]} \mu_N \Phi_N(j\tau)(\{\|\phi^N\|_{\mathcal{H}} > K\}) = 2[T/\tau]\mu_N(\{\|\phi^N\|_{\mathcal{H}} > K\})$$

  $$\implies \quad \mu(\Omega_N^c) \lesssim \frac{T}{\tau}\mu_N(\{\|\phi^N\|_{\mathcal{H}} > K\}) \sim TK^C e^{-cK^2} \qquad \text{(Fernique-tail).}$$

  $\implies$ By choosing $K \sim \left(\log \frac{T}{\varepsilon}\right)^{\frac{1}{2}}$, we have $\mu(\Omega_N^c) < \varepsilon$.

- By its construction, $\|u^N(j\tau)\|_{\mathcal{H}} \leq K$ for $j = 0, \cdots, \pm[T/\tau]$.
  $\implies$ By local theory,

  $$\|u^N(t)\|_{\mathcal{H}} \leq 2K \sim \left(\log \frac{T}{\varepsilon}\right)^{\frac{1}{2}} \text{ for } |t| \leq T.$$

Extending u globally in time

Recall $\mathcal{H} = H^s(\mathbb{T})$, $s < \frac{1}{2}$ fixed.

Lemma (Approximation Lemma)

Let u^N be the solution to FDA with initial data $\phi^N = P_N(\phi), \phi \in \mathcal{H}$. Assume the a priori bound

$$\|u^N(t)\|_{\mathcal{H}} \leq A, \qquad \text{for all } t \in [0, T].$$

Then the solution u to the NLS initial value problem with data ϕ satisfies for any $s_1 < s$,

$$\|u(t) - u^N(t)\|_{H^{s_1}(\mathbb{T})} < e^{C_{s,A}\, T} N^{s_1 - s},$$

which tends to 0 as $N \to \infty$.

We use u^N to walk u to its side pass the local time τ and up to time T.

The proof is it is not hard but not entirely trivial because we cannot compare u and u^N directly on $[0, T]$: after the first step on $[0, \tau]$, u and u^N may in principle start becoming apart: we have no a priori bound on u on $[0, T]$ –which is the whole point.

One considers as a stepping stone the solution u' to:

$$(SS) \qquad \begin{cases} iu'_t + \Delta u' + |u'|^4 u' = 0 \\ u'(0, x) = \phi^N(x) := P_N \phi(x) = \sum_{|n| \leq N} \widehat{\phi}(n) e^{inx}, \qquad x \in \mathbb{T}. \end{cases}$$

and compare u to u' and u' to u^N -which have same truncated initial data– in an iterative argument on each $[j\tau, (j+1)\tau], \ j = 0, \dots [\frac{T}{\tau}]$ and piecing them together to obtain the desired bound.

Here τ is the local existence time for NLS with data bounded in H^{s_1} by $A + 1$; ie.

$$\tau \sim \frac{C_{s_1}}{(1 + \|\phi\|_{H^{s_1}})^{c_{s_1}}}$$

1. $$\|u(0) - u'(0)\|_{s_1} < N^{s_1 - s} A$$
2. Assume that for $t \leq t_0$ we obtained

   $$\|u(t) - u^N(t)\|_{s_1} < \delta < 1$$

   then

   $$\|u(t_0)\|_{s_1} \leq \|u^N(t_0)\|_{s_1} + \delta < A + 1$$

   by our a priori bound hypothesis.
3. From the local theory, the IVP for the NLS equation with data $u(t_0)$ at $t = t_0$ and for the SS equation with data $u'(t_0) = u^N(t_0)$ are well posed on $[t_0, t_0 + \tau]$. Moreover, for $t \leq t_0 + \tau$,

   $$\|u(t) - u'(t)\|_{s_1} \leq 2\|u(t_0) - u'(t_0)\|_{s_1} < 2\delta$$

   by (2) above.

Next, we want to compare $u'(t)$ and $u^N(t)$ on $[t_0, t_0 + \tau]$. Since they have the same initial data,

$$u'(t) - u^N(t) = c \int_0^t S(t - t')\Gamma(t')\, dt'$$

where

$$\begin{aligned} \Gamma(t') &= |u'|^4 u' - P_N\big(|u^N|^4 u^N\big) \\ &= \big[|u'|^4 u' - P_N\big(|u'|^4 u'\big)\big] + \Big[P_N\big(|u'|^4 u'\big) - P_N\big(|u^N|^4 u_N\big)\Big] \end{aligned}$$

Do a fix point argument on X^{s_1} as in the local theory but now thanks to the choice of τ, the a priori bound and the steps above we obtain:

$$\|u'(t) - u^N(t)\|_{s_1} < CAN^{s_1 - s} \qquad t \in [t_0, t_0 + \tau]$$

whence combining with (2) above gives that

$$\|u(t) - u^N(t)\|_{s_1} < 2\delta + CAN^{s_1 - s}.$$

At this point by the choice of τ we can iterate the argument on each time subinterval

$$[j\tau,\, (j+1)\tau], \qquad j = 0, \dots [\frac{T}{\tau}]$$

to obtain a recursive estimate

$$\|u(j\tau) - u^N(j\tau)\|_{s_1} < \delta_j \,+\, CAN^{s_1 - s},$$

$$\delta_j < C^{j+1} A\, N^{s_1 - s}$$

which yields the conclusion provided N is large enough so that $C^{j+1} N^{s_1 - s} < 1$.

So far: we have μ_N invariant $\Longrightarrow u^N$, u extends globally for "good" data and that u^N converging uniformly to u.

We turn then to μ, which we want to realize as a weighted Gaussian measure

$$" \quad d\mu \,=\, \tilde{Z}^{-1}\, \chi_{\|\phi\|_{L^2} \leq B}\; e^{\int |\phi|^6\, dx}\; d\rho \quad "$$

and the weak limit of the finite dimensional μ_N.

- Recall ρ yields for ϕ the distribution of a random (Fourier) series

  (1.1) $$\phi = \phi^\omega = \sum_{n \in \mathbb{Z}^d} \frac{g_n(\omega)}{\sqrt{1+|n|^2}} e^{in \cdot x}.$$

  which defines a.s. a function in $H^s(\mathbb{T})$ for $s < \frac{1}{2}$.
  ($\widehat{\phi}_n = \frac{g_n}{\sqrt{1+|n|^2}}$ on the support of ρ.)
- The existence a.s. of $\int |\phi^\omega|^6 dx$ follows from the fact that $\phi^\omega \in H^s(\mathbb{T}), s < \frac{1}{2}$ a.s.

Proposition (Lebowitz-Rose-Speer)

The weight

$$\chi_{\|\phi\|_{L^2}\le B}\ e^{\int |\phi|^6\,dx} \in L^r(d\rho)$$

for all $1 < r < \infty$ *provided* $B = B(r)$ *is sufficiently small.*

Idea of Bourgain's proof: For any $\lambda > 0$ fixed prove that

$$\mathbb{P}_\omega\left[\|\sum \frac{g_n(\omega)}{\langle n\rangle} e^{inx}\|_{L^6} > \lambda,\ \Big(\sum|\frac{g_n(\omega)}{\langle n\rangle}|^2\Big)^{1/2} \le B\right] \lesssim e^{-c\lambda^2(\frac{\lambda}{B})^4}$$

by a suitable Littlewood-Paley decomposition and a large deviation estimate on each dyadic block. Then,

$$\begin{aligned}
&\|\chi_{\|\phi\|_{L^2}\le B}\ e^{\int |\phi|^6\,dx}\|_{L^r(d\rho)} \\
&\le \int_{\|\phi\|_{L^6}\le\lambda} e^{r\int|\phi|^6 dx}\chi_{\|\phi\|_{L^2}\le B}\,d\rho\ + \\
&\qquad + \sum_{j=1}^{\infty}\int_{2^j\lambda\le\|\phi\|_{L^6}\le 2^{j+1}\lambda} e^{r\int|\phi|^6 dx}\chi_{\|\phi\|_{L^2}\le B}\,d\rho
\end{aligned}$$

The latter is less than or equal to

$$e^{r\lambda^6} + \sum_{j=1}^{\infty} e^{(r2^{j+1}\lambda)^6} \rho\big(\|\phi\|_{L^6} \geq 2^j\lambda,\ \|\phi\|_{L^2} \leq B\big)$$

which by Bourgain estimate above, is finite provided B is sufficiently small.

Something smaller than $(\frac{C}{64r})^{1/4}$ would do.

We then have:

Theorem (Existence of Gibbs measure)

We now have that for small B, the measure μ is a probability measure which is absolute continuous with respect to the Gaussian measure ρ and μ_N converges weakly to μ.

If $E_N := \mathrm{span}\{\mathrm{e^{inx}} : |\mathrm{n}| \leq \mathrm{N}\}$, and U is an open in $H^s(\mathbb{T}), s < 1/2$, one has:

$$\rho(U) = \lim_{N\to\infty} \mu_N(U \cap E_N), \qquad \mu(U) = \lim_{N\to\infty} \mu_N(U \cap E_N).$$

Almost sure GWP for NLS

Combining the results above we obtain:

Theorem

For any given $T > 0$ and $\varepsilon > 0$ there exists a set $\Omega(\varepsilon, T)$ such that

(a) $\mu\,(\Omega(\varepsilon, T)) \geq 1 - \varepsilon$.

(b) For any initial condition $\phi \in \Omega(\varepsilon, T)$ the IVP for NLS is well-posed on $[-T, T]$ with the bound

$$\sup_{|t|\leq T} \|u(t)\|_{H^{\frac{1}{2}-}(\mathbb{T})} \lesssim \left(\log \frac{T}{\varepsilon} \right)^{\frac{1}{2}}.$$

Now, a standard argument, by which for fixed $T > 0$ we first take $\varepsilon = 2^{-i}$, and let $\Omega_T := \cup_i \Omega(2^{-i}, T)$, $\mu(\Omega_T) \geq 1 - 2^{-i}$ and then let $T = 2^j$ and $\Omega = \cap_j \Omega_{2^j}$ gives:

Theorem [Almost sure global well-posedness of NLS]

There exists a subset Ω of $H^{\frac{1}{2}-}(\mathbb{T})$ with $\mu(\Omega^c) = 0$ such that for every $\phi \in \Omega$, the IVP for NLS with initial data ϕ is globally well-posed.

From here, now that we have a well defined global NLS flow on the support of the Gibbs measure μ, its invariance follows from standard argument (Zhidkov, Bourgain).

Theorem 2 [Invariance of μ]

The measure μ is invariant under the flow $\Phi(t)$ of (GDNLS)