$\forall u_0 \in H$, $\exists!$ $u(t,u_0)$ soln in $C(C[0,T]; H)$, $\forall T$

$$P_t \varphi(u_0) = \mathbb{E}(\varphi(u(t,u_0)))$$

$(\nabla P_t) = P_t^* \nu = \nu$ $\quad \exists$ follows from tightness of $\frac{1}{t}\int_0^t \mu_s \, ds$

Uniqueness (~~an~~ case)

$du = (Au - u^3)\, dt + \varphi\, dw$, Dirichlet bdy

φ st $\exists!$ global soln $(\varphi = I)$

2 invariant measures ν_1, ν_2

u_0^1, u_0^2 law ν_1, ν_2

$\to u^1, u^2$

$r = u^1 - u^2$

$\frac{dr}{dt} = AR - (u^1)^3 - (u^2)^3$

$\frac{1}{2} \frac{d|r|^2}{dt} + |r_x|^2 \leq 0$

$\frac{1}{2} \frac{d|r|^2}{dt} + \lambda_1 |r|^2 \leq 0$

$\underline{\mathbb{E}|r|^2} \leq e^{-2\lambda_1 t}\, \underline{\mathbb{E}|u_0^1 - u_0^2|}$

$\mathbb{E}(\varphi(u^1(t))) - \mathbb{E}(\varphi(u^2(t))) \to 0$

$\downarrow \int \varphi\, d\nu_1$ $\quad \downarrow \int \varphi\, d\nu_2$

if φ lipshitz

$\to \nu_1 = \nu_2$

Kolmogorov eqn: $du = (Au + f(u))dt + \varphi\, dW$

f: Burgers, R.D or N.S, or

$\Psi(t,u_0) = \mathbb{E}(\varphi(u(t,u_0))) = P_t \varphi(u_0)$

Ito Formula: $d\varphi(u(t,u_0)) = (Au + f(u), D\varphi(u(t,u_0))) + (D\varphi(u(t,u_0)), \phi\, dw) + \frac{1}{2}\operatorname{Tr} \phi\phi^* D^2\varphi(u(t,u_0))$

Take $\mathbb{E} \to t=0$ and then t_0

Kol. eq^n $\to \begin{cases} d\psi/dt = \frac{1}{2}\operatorname{Tr} \phi\phi^* D^2\psi + (Au + f(u), D\varphi) = L\psi \\ \psi(0) = \varphi \end{cases}$

Much more than for $\phi = 0$ because of $\operatorname{Tr} \phi\phi^* D^2\psi$

- $\phi = I$: "$\operatorname{Tr} D^2\psi = \Delta\psi$"

$\to$ uniqueness is easier.

- ϕ degenerate $\longrightarrow$ rank $\phi < \infty$

$\to$ hypoelliptic case

In finite dimension, the adjoint of the K eq.

$\to$ Fokker-Planck eq.

In infinite dimension:

$\to$ no Lebesgue measure

gradient system:

recall

ex: reaction diffusion: $\to V(u) = 1/4 \int |u|^4 d\xi$, $DV(u) = u^3$

$$du = (Au + DV(u))\,dt + dW$$

$(A = \Delta + \text{Dir bdy cond})$

$$L\psi = \frac{1}{2}\operatorname{Tr} D^2\psi + (Au + DV(u), D\psi)$$

$\frac{1}{2}\int L\psi\, e^{-V(u)}\, d\nu(u)$	"$\frac{1}{2}\int \operatorname{Tr} D^2\psi\, e^{-V(u)} e^{-\frac{1}{2}\lvert A^{\frac{1}{2}} u\rvert^2} du$"
$\nu = N(0, \frac{1}{2}(-A)^{-1})$	$= -(D\psi(u), V'(u)) - (D\psi(u), Au)$
inv measure for $V = 0$	

$\int L\phi \, e^{-V(u)} d\nu = 0$

$d/dt \int P_t \, \phi(u) e^{-V(u)} d\nu = \int \phi d\mu \qquad \mu = \frac{1}{Z} e^{-V} \nu$

$$= \int L P_t \phi(u) e^{-V(u)} d\nu \Big\} \Rightarrow \mu = \frac{1}{Z} e^{-V} \nu \quad \text{invariant measure}$$

$= 0$

6) Rough, but not too rough, noise

Reaction diffusion with space-time white noise

$$\begin{cases} du = (\Delta u + u^3 - u)dt + dW \\ \text{Periodic bdy condition on } [0,1]^d \end{cases}$$

$d=1$ easy

$d=2$ critical $\quad dz = Az + dW \qquad z \in C([0,T], H^s) \ \forall s>0$

$d=3$: rough $\to$ MH

In fact $z \in C([0,T], B^{s}_{p,r})$

$s<0, \ p, r \geq 1$

$u = y + z \qquad A = \Delta - I \qquad dz = Az + dW$

$dy/dt = Ay + (y+z)^3 = Ay + y^3 + 3y^2 z + 3y z^2 + z^3$

z is gaussian: $z \sim N(0, \frac{1}{2}(-A)^{-1})$

$z \notin H, \ Tr(-A)^{-1} = \infty$

Up to renormalization

z^p can be defined

- where this renormalized z^p lives?
- is y sufficiently smooth to define yz^2 and $y^2 z$

Gaussian calculus, Wick products

Facts: • $\int_{H^{-1}} (x, (-A)^{1/2} y)^2 \, d\mu(x) = |y|_H^2 = \mathbb{E}(z, (-A)^{-1/2} y)^2$

$x \rightsquigarrow (x, (-A)^{1/2} y)$

exists in $L^2(H^{-1}, \mu)$ for $y \in H$

• $\int e^{(x, (-A)^{1/2} y)} \, d\mu(x) = e^{|y|_H^2/2}$

Hermite polynomials.

$$e^{-t^2/2 + t\xi} = \sum_0^\infty \frac{t^m}{\sqrt{m!}} H_m(\xi)$$

$$\int H_n((x, (-A)^{1/2} y)) H_m((x, (-A)^{1/2} \tilde{y})) \, d\mu = \delta_{n,m} (y, \tilde{y})^n$$

$$V_n = \mathrm{Sp}\{ x \rightsquigarrow H_n(x, (-A)^{1/2} y), |y| = 1 \}$$

$$L^2(H^{-1}, \mu) = \bigoplus_{\perp, n \in \mathbb{N}} \underbrace{V_n}_{n^{\text{th}} \text{ Wiener class}}$$

$z^3(\xi) \qquad \xi \in (0,1)$

$"(z, \xi)^3" = \lim_{\varepsilon \to 0} (z, \rho_\varepsilon(\cdot - \xi))$

$\to x \rightsquigarrow (x, (-A)^{1/2} y)^3$

with $y = (-A)^{-1/2} \rho_\varepsilon(\cdot - \xi)$

$$\underset{\text{proj onto } V_3}{\Pi_3} (x \rightsquigarrow (x, (-A)^{1/2} y)^3) = \sqrt{3}\, H_3(x \rightsquigarrow (x, (-A)^{1/2} y)^3)$$

for $|y| = 1$

$$z_\varepsilon^3(\xi) = (z, \rho_\varepsilon(\cdot-\xi))^3 = K_\varepsilon^3\,(z, (-A)^{1/2}\tilde\rho_{\varepsilon,\xi})$$

$$\tilde\rho_{\varepsilon,\xi} = 1/K_\varepsilon\,(-A)^{-1/2}\rho_\varepsilon(\cdot-\xi)$$

$$K_\varepsilon = |(-A)^{-1/2}\rho_\varepsilon(\cdot-\xi)|_H \xrightarrow[\varepsilon\to 0]{} \infty$$

$$\Pi_3(z_\varepsilon(\xi)^3) = K_\varepsilon^3\sqrt{6}\cdot H_3\left(\frac{z_\varepsilon(\xi)}{K_\varepsilon}\right) = z_\varepsilon(\xi)^3 \underbrace{-3K_\varepsilon z_\varepsilon(\xi)}_{\text{renormalization}}$$

If z has low then

$\Pi_3(z_\varepsilon(\xi)^3)$ converges

$\Pi_3(z_\varepsilon^3)$ converges in $L^2(\Omega, H^s)$ $\forall s<0$
and in $L^k(\Omega, B^s_{p,r})$ $\forall s<0$ ← Besov space

$\to$ $:z^3: := \lim_{\varepsilon\to 0}\Pi_3(z_\varepsilon^3) \in L^k(\Omega, B^s_{p,r})$ $\forall s<0,\ k,p,r\geq 1$

Something for $:z^2:$

$$dy/dt = Ay + y^3 + 3y^2 z + 3yz^2 + z^3$$

$$\to\ dy/dt = Ay + y^3 + 3y^2 :z: + 3y\underbrace{:z^2:}_{} + \underbrace{:z^3:}_{} \to B^k_{p,r}$$

Look for $y \in C([0,T], B^s_{p,r}) \cap L^\alpha(0,T, B^s_{p,r})$
for well chosen α, β, and p large

- $(v^1, v^2) \longrightarrow v^1 v^2$

  $B^a_{p,r}\times B^b_{p,r} \longrightarrow B^{a+b-2/p}_{p,r}$ $\quad a+b>0$ if $a,b<2/p$

- $t \rightsquigarrow\ :z(t)^3:$ $\quad \int_0^T \mathbb{E}(|:z(t)^3:|^k_{B^s_{p,r}}) = C_{p,r,k,s}T$

$:Z^3: \in L^K(0,T, B^s_{p,r})$

$\rightarrow \forall u_0 \in B^s_{p,r}, \quad \exists! u \in C([0,T], B^s_{p,r})$
$T = T^*(u)$

$\nu(dx) := \frac{1}{2} e^{-\int :x^4: d\xi} \mu(dx)$ invariant measure

$\longrightarrow$ global existence for ν a.e. u_0

Mourrant-Weber - proved global existence for every u_0
(not just with inv measure)

- First argument of step 0 of MH.