GRIDS WITH DENSE ORBITS

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1. Introduction

Let $X_d = \mathrm{SL}(d,\mathbb{R})/\mathrm{SL}(d,\mathbb{Z})$ be the space of unimodular lattices. We will be interested in inhomogeneous approximation and hence will study *grids*, affine shifts of lattices. We can consider these as the space $Y_d = \mathrm{ASL}(d,\mathbb{R})/\mathrm{ASL}(d,\mathbb{Z})$. There is a well defined projection $\pi : Y_d \to X_d$. The fiber over $x \in X_d$ is the torus $\pi^{-1}(x) = \mathbb{R}^d/x$. As $x \to \infty$ in X, the tori become more and more degenerate.

Fix some function $F : \mathbb{R}^d \to \mathbb{R}$.

Question. What values does F take on lattices/grids?

Definition 1 (Value set)**.**

$$V_F(y) = \{F(w) \mid w \in y\}$$

Definition 2. $y \in Y_d$ is said to be a *dense value grid*, DV_F, if $\overline{V_F(y)} = F(\mathbb{R}^d)$.

Definition 3. $X \in X_d$ is a.s. *grid-DV_F* if for a.e. $y \in \pi^{-1}(x)$, y is DV_F. More generally, given a probability measure μ on $\pi^{-1}(x)$, we say that x is *μ-a.s. grid DV_F* if μ-a.e. $y \in \pi^{-1}$ is DV_F.

Examples.

(1) $F(v) = N(v) = \left|\prod_{i=1}^d V_i\right|$
(2) $F(v) = P(v) = \|(v_1,\ldots,v_{d-1})\| \cdot |v_d|^{\frac{1}{d-1}}$
(3) $F =$ independent quadratic form

Note. If $d = 2$ all three examples are the same.

2. Dynamcs

Definition 4. The *invariance group* of F is

$$H_F = \{g \in \mathrm{SL}(d,\mathbb{R}) \mid f \circ g = F\}^\circ$$

Examples. For the examples above, the invariance groups are

(1) $H_F = A$, the full diagonal group
(2) $H_F = a(t) = \mathrm{diag}(e^t,\ldots,e^t,e^{-(d-1)t})$, up to compact group
(3) $H_F = \mathrm{SO}(F)$

Standing assumption: H_F is non-compact.

Observation 1. Given a grid $y_0 \in \overline{H_F y}$, then $\overline{V_F(y_0)} \subset \overline{V_f(y)}$. In particular if y_0 is DV_F, then so is y.

By Howe-Moore theorem, we get that H_F acts ergodically on Y_d and so a.e. grid y is DV_F. It follows that almost any x in X_d is almost surely grid DV_F.

Goal. *Develop a better understanding of the set of a.s. grid DV_F lattices. In particular, we would like to develop a dynamical criterion to determine whether a given lattice is is a.s. grid DV_F.*

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Assume x has a non-diverget H_F orbit, then x accumulates on x_0 by a sequence in the invariance group. Consider the fibers above x and x_0, we want to know what the possible accumulation points of a.e. grids in the $\mathbb{R}^n/x$ are in $\mathbb{R}^n/x_0$. If there are many accumulation points, x might be DV_F.

What happens when the orbit is divergent? Let $F = N$, $H_F = A$, $x = \mathbb{Z}^d$, and $A\mathbb{Z}^d$ is divergent then

Exercise 1. Show that for all $w \in \mathbb{R}^d$ we have

$$V_N(+ w) = \left\{ \left| \prod_{i=1}^{d} (m_i + w_i) \right| \mid m_i \in \mathbb{Z} \right\}$$

is discrete for all w.

3. Results

3.1. Two hypotheses (on F) and main theorem.

H1: For any $y \in Y_d$ and any line $U \in \mathbb{R}^d$, we have $\overline{F(y+U)} = \overline{F(\mathbb{R}^d)}$

H2: For all $h_n \in H_F$, with $h_n \to \infty$ we have that ll the eigenvalues of h_n go to ∞ or 0.

Note. N satisfies H1, P satisfies H2 but not H1.

Theorem 1 (Shapira)**.** *Let F be a nonimpact invariance group and let x be a lattice with non-divergent H_F orbit and assume either H1 or H2, then x is a.s. grid DV_F.*

Question 1. For which measures μ on $\pi^{-1}(x)$ does the result still hold?

3.2. Application: $F = N$.

Conjecture 1 (Minkowski)**.** *For all $y \in Y_d$, $V_N(y) \cap [0, 2^{-d}] \neq \emptyset$.*

Theorem 2. *For all $x \in X_d$*

$$\lambda_{\pi^{-1}}(x) \left(\left\{ y \in \pi^{-1}(x) \mid y \textit{ violates the conjecture} \right\} \right) = 0$$

Bomber ('64): used Fourier techniques to show that this measure $\leq 1 - 2^{-\frac{d+1}{2}}$

3.3. Application: $F = P$.

For $v \in \mathbb{R}^{d-1}$ let $x_v = \begin{pmatrix} I_{d-1} & v \\ 0 & 1 \end{pmatrix} \cdot \mathbb{Z}^d$. We call v *non-singular* if $H_P x_v$ is non-divergent. Applying the theorem we get

Corollary 1. *For all v non-singular and a.e. $w \in \mathbb{R}^{d-1}$,*

$$\liminf_n |n|^{\frac{1}{d-1}} \cdot \|nv - w\| = 0$$

Example 1 (Kim-Tseng)**.** For $d = 2$, v is non-singular if and only if it is irrational.

Problem 1 (Open)**.** Is it true that if $a(t)x$ is non-divergent then x is μ a.s. grid DV_F with respect to any coset Haar measure on $\pi^{-2}(x)$.