ERGODIC PROPERTIES OF HOROCYCLIC FLOWS ON INFINITE VOLUME HYPERBOLIC SURFACES

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1. Horocyclic flow

1.1. **Geometric view.** In the disc model, $\mathbb{D}$, of hyperbolic space, geodesics are circular arcs which meet the boundary at right angles. The *geodesic flow*, g^t, translates a vector v along the geodesic. Horocycles are circles in the interior of the disc that are tangent to the boundary. The *horocyclic flow*, h^s, moves the vector v along the horocycle, as shown in Fig. 1.

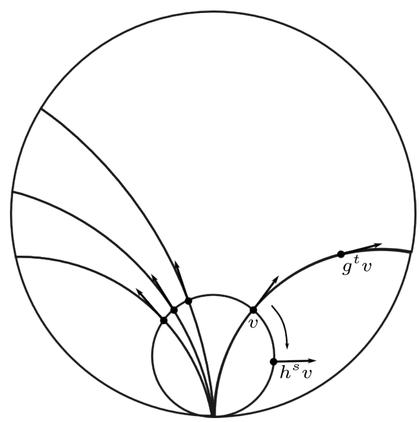

Figure 1. Geometric picture of the geodesic and the horocyclic flows

1.2. **Group view.** Let $G = \mathrm{PSL}(2,\mathbb{R}) \simeq T^1\mathbb{D}$. Then

- The *geodesic flow* acts by multiplication on the right by the matrix $\begin{pmatrix} e^{t/2} & \\ & e^{-t/2} \end{pmatrix}$
- The *horocyclic flow* acts by multiplication on the right by the matrix $\begin{pmatrix} 1 & \\ s & 1 \end{pmatrix}$

We have the fundamental relation $g^t h^x = h^{se^t} g^t$, and the following description of the unstable foliation:

$$(h^s v)_{s\in\mathbb{R}} = W^{su}_{g^t}(v) = \left\{ w, d(g^{-t}w, g^{-t}v) \xrightarrow[t\to\infty]{} 0 \right\}$$

In this way we see that the ergodic behavior of $(h^s v)$ is related to the asymptotic behavior of $(g^{-t}v)$ as $t \to \infty$.

Let $\Gamma < G$ be a discrete group, set $S = \Gamma\backslash\mathbb{H}$ then $T^1S = \Gamma\backslash G$ and have right actions of (g^t) and (h^s) on this space.

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1.3. **Goal.** Study the ergodic properties of (h^s) when $\mathrm{vol}(S) = +\infty$, and Γ is non-elementary (i.e. S is not the hyperbolic plane).

Suppose $\Gamma\backslash G$ has finite volume, then we have the following results

- (Hedlund) All (h^s) orbits are either periodic or dense.
- (Furstenburg, Dani) Except for periodic measures, the Liouville measure, $\mathcal{L}$ is the unique (h^s)-invariant ergodic measure.

2. Basic ergodic theory results

2.1. $\mu(X) < \infty$.

Theorem 1 (Birkhoff ergodic theorem). *Consider the space (X, g^t, μ) where μ is an invariant ergodic probability measure. Then for any $\phi \in L^1(\mu)$,*

$$\frac{1}{T}\int_0^T \phi(g^t x)\,dt \to \frac{1}{\mu(X)}\int \phi d\mu$$

This theorem does not say anything specific for a given $x \in X$.

If μ is unique and X compact, then convergence holds for all x.

Theorem 2 (Dani-Smillie). *For all v non-periodic and all $\phi \in C_c(T^1S)$,*

$$\frac{1}{T}\int_0^T \phi(h^s v)\,ds \to \int \phi d\mathcal{L}$$

2.2. **Infinite ergodic theory.**

Definition 1. Let μ be an invariant ergodic measure for g^t, and $\mu(X) = \infty$, then we call μ *conservative* if for all A with $\mu(A) > 0$, μ-a.e $x \in A$ satisfies

$$\int_0^\infty \mathbb{1}_A(g^t x)\mathrm{d}t = +\infty$$

Remark 1. The Birkhoff ergodic theorem is true in the infinite case and we have

$$\frac{1}{T}\int_0^T \phi(g^t x)\mathrm{d}t \to 0$$

Theorem 3 (Hopf ergodic theorem). *Let μ be an ergodic and conservative measure and $\phi, \psi \in L^1(\mu)$, then for μ-almost every $x \in X$,*

$$\frac{\int_0^t \phi(g^t x)\,dt}{\int_0^T \psi(g^t x)\,dt} \to \frac{\int_X \phi\, d\mu}{\int_X \psi\, d\mu}$$

As in the case with the Birkhoff ergodic theorem, given a specific $x \in X$ we do not understand what happens with these ratios.

In the finite measure case we have

$$\int_0^T \phi(g^t x)\mathrm{d}t \sim \frac{T}{\mu(X)}\int \phi \mathrm{d}\mu,$$

we would like to find a similar equivalence for the infinite case, unfortunately one does not exist.

Theorem 4 (Aaronson). *Suppose $a(T) \to \infty$, then μ-almost surely either* $\limsup \frac{1}{a(T)}\int_0^T \phi(g^t x)\,dt = +\infty$ *or* $\liminf \frac{1}{a(T)}\int_0^T \phi(g^t x)\,dt = 0$.

3. Results in the infinite volume case

Remark 2. Ergodic measures for $(h^s v)$ give a "way" for (g^{-t}) to go to infinity

Theorem 5 (Roblin)**.** *If (g^t) admits a finite measure of maximum entropy then there is a unique (h^s)-invariant ergodic measure supported on the set*

$$\left\{v \in T^1 S \mid (g^{-t}v)_{t\geq 0} \text{ returns infinitely often to a compact set}\right\}$$

A surface S is *geometrically finite* if it has finitely many cusps and finitely many funnels. In dimension 2, this condition is equivalent to Γ being finitely generated. In this case, if a vector v enters the funnel, it will never come back and its horocyclic orbit will be closed and non-periodic. If a vector w negatively points into the cusp, then it's horocyclic orbit will be periodic. All other horocyclic orbits are dense.

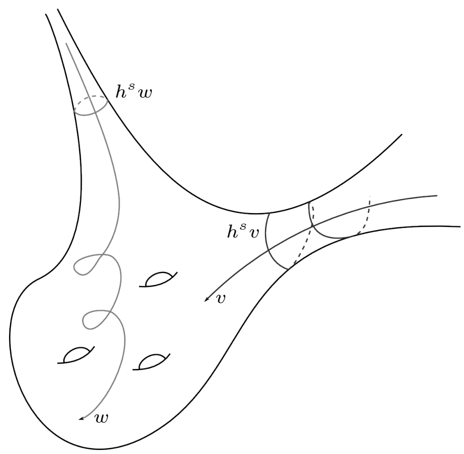

Figure 2. Examples of types of horocyclic orbits on a geometrically finite surface.

Corollary 1 (Burger/Roblin)**.** When S is geometrically finite there exists a unique invariant ergodic (h^s) measure m_{BR} whose support is exactly the set $\{v \mid (g^t v) \text{ does not go into the funnel}\}$.

Theorem 6 (Maucourant - Schapira)**.** *Let S be geometrically finite with critical exponent $0 < \delta < 1$. Then for all v, non (h^s)-periodic, such that $(g^t v)$ does not go into the funnel and for all $\phi \in C_c(T^1 S)$ we have*

$$\int_t^t \phi(h^s v)\, ds \sim t^\delta \tau(v,t) \int \phi \, dm_{BR}$$

where $\tau(v,t)$ is a continuous, well understood, bounded function.

Example 1. Let $\Gamma \lhd \Gamma_0$ where Γ_0 is a cocompact lattice and $\Gamma_0/\Gamma \simeq \mathbb{Z}^d$. Let $S = \Gamma\backslash\mathbb{H}$ be a surface with no cusps and $S_0 = \Gamma_0\backslash\mathbb{H}$. Consider the vector v and look at it's negative orbit. At time t it is somewhere in $\mathbb{Z}^d$, see Fig. 3.

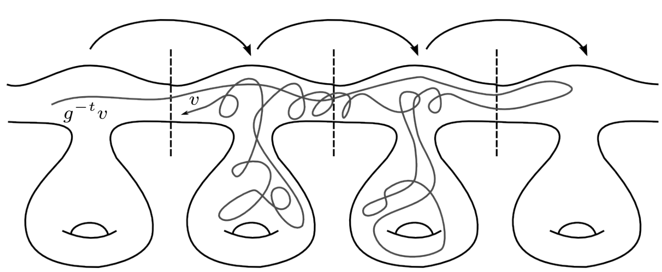

Figure 3. The vector v in S and its negative orbit

If ζ is a coordinate in $\mathbb{Z}^d$ then almost surely,

$$\frac{1}{t}\xi(g^{-t}v) \xrightarrow[t\to\infty]{} \zeta_\infty(v)$$

and we call $\xi_\infty(z)$ the *asymptotic cocycle*, which gives us an asymptotic speed in $\mathbb{Z}^d$. Note that if m is ergodic, this function is constant almost surely, and we can call this value $\xi_\infty(m)$.

$C = \{\xi_\infty(v)\}$ is a compact, convex set and the interior, $\mathring{C}$, is diffeomorphic to the unit ball of $\mathbb{R}^d$ for a certain "good" norm.

Theorem 7 (Babillot - Ledrappier, Sarig)**.** *The function $m \mapsto \xi_\infty(m)$ is a bijection between the set of (h^s)-invariant ergodic measures and the set $\mathring{C}$.*

There is an associated equidistribution result:

Theorem 8 (Sarig - Schapira)**.** *For all v, $\xi_\infty(v) = \xi_\infty(m)$ if and only if $(h^s v)$ is equidistributed with respect to m.*

4. Tools

Assume $\Gamma < \mathrm{PSL}(2,\mathbb{R})$ be finitely generated and discrete. Let $\Lambda_\Gamma = \overline{\Gamma_0} \setminus \Gamma_0 \subset S^1$ be its *limit set* If $\mathrm{vol}(S) < \infty$, then $\Lambda_\Gamma = S^1$ and if $\mathrm{vol}(S) = \infty$ then Λ_Γ can be a Cantor set.

We can give coordinates to a geodesic $v \in \mathbb{D}$ by its endpoints so that $T^1\mathbb{D} \simeq S^1 \times S^1 \setminus \mathrm{diag} \times \mathbb{R}$ where we write (v^-, v^+, s) for the point corresponding to $v \in T^1\mathbb{D}$. Then in these coordinates, the geodesic flow acts by translation, $(g^t v) = (v^-, v^+, s+t)$, and the horocyclic flow fixes the first and last coordinate but moves the positive endpoint, $(h^s v) = (v^-, w^+, s)$.

$T^1 S = \Gamma\backslash(S^1 \times S^1 \times \mathbb{R})$. The non-wandering set of the geodesic flow is the set vectors whose endpoints are in the limit set, $\Omega = \Gamma\backslash(\Lambda_\Gamma \times \Lambda_\Gamma \times \mathbb{R})$. This set is not invariant under the horocyclic flow. We definite the non-wandering set of (h^s) to be

$$\mathcal{E} = \bigcup_{s\in\mathbb{R}} h^s\Omega = \Gamma\backslash(\Lambda_\Gamma \times S^1 \times \mathbb{R})$$

We define the constant $\delta_\Gamma = \limsup_{R\to\infty} \frac{1}{R} \log \#\{\gamma \in \Gamma, d(0,\gamma) \leq R\}$. In the finite dimensional case, $\delta_\Gamma = \dim_H \Lambda_\Gamma = h_{\mathrm{top}}(g^t)$, the Hausdorff dimension of the limit set and the topological entropy of the geodesic flow.