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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Sean Howe Email/Phone: seanpkh@gmail.com

Speaker's Name: Frank Calegari

Talk Title: The Bloch-Kato conjecture for some adjoint Selmer groups

Date: 12 / 4 / 2014 Time: 11 : 00 am / pm (circle one)

List 6-12 key words for the talk: Bloch-Kato, Adjoint Selmer Groups, R=T, modularity lifting

Please summarize the lecture in 5 or fewer sentences: Describes the relation between modularity theorems and results on Bloch-Kato. Describes new results on modularity lifting ~~[illegible]~~ for GSp(4) and applies them to prove Bloch-Kato type results for genus 2 curves.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
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- Joint w/ D. Geraghty
- Joint w/ D. Geraghty, M. Harris

Let K be a #-field of sig. (r_1, r_2)

$$\lim_{s \to 1^+} (s-1)\zeta_K(s) = \frac{2^{r_1}(2\pi)^{r_2} h_K R_K}{w_K \cdot \sqrt{|\Delta_K|}}$$

$$\lim_{s \to 0} s^{-r}\zeta_K(s) = \frac{-h_K \cdot R_K}{w_K}$$

$$r = \operatorname{rank} \mathcal{O}_K^\times$$

Pure motive M

Relate
1. Leading term of $L^*(M, K)$
2. A regulator
3. Order of class group
4. Transcendental factors

Replace M by V = p-adic Galois rep.
- rank/regulators
- transcendental factors
- Selmer groups

Let V be a representation of $G_{\mathbb{Q},S}$

$H^1(\mathbb{Q}_S, V)$ classifies $0 \to V \to W \to \mathbb{Q}_p \to 0$

Ex 1) If $V = \mathbb{Q}_p$, $H^1(\mathbb{Q}_S, \mathbb{Q}_p) \cong \mathbb{Q}_p$

(unique $\mathbb{Z}_p$ extension of $\mathbb{Q}$, cyclo)

2) Let $E/\mathbb{Q}$ be an elliptic curve

$E[p]$ irred as a Galois mod

$P \in E(\mathbb{Q})$, P of ∞-order, not div. by p

$Q_n := \frac{1}{p^n} P$ generates

$$0 \to E[p^n] \to W_n \to \mathbb{Z}/p^n\mathbb{Z} \to 0$$

$$0 \to V_p E \to W \to \mathbb{Q}_p \to 0 \quad \text{in } H^1_{\mathbb{Q}_S}(V_p(E))$$

W^* can be realized in cohomology of $E - \{P, \infty\}$.

Calegari 2

$H^1_f(\mathbb{Q}_S, V) =$ Selmer group
$\sim$ appropriate local property at p to come from geometry.

V is "geometric" and irreducible

F. M. $\Rightarrow$ V comes from geometry. Ask to compute
$\dim H^1_f(\mathbb{Q}_S, V)$.

<u>Conj</u> For any pure, irred. V, $\mathrm{Ext}^1_f(V, V) = 0$
$\left(\text{i.e. } H^1_f(\mathbb{Q}_S, \mathrm{Ad}\, V) = 0\right)$

$$\frac{L(\pi \times \pi^\vee, 1)}{\zeta(1)}$$ related to this via Bloch-Kato

Let $X/\mathbb{Q}$ be a curve of genus 1, $A = \mathrm{Jac}\, X$, assume A is modular
(A semistable)

$$V = T_p(A) \qquad W = \mathrm{Ad}^0(V)$$
$$\cup\!\shortmid$$
T integral lattice

$\phi : X_0(N) \to A$

Shimura: $X_0(N) \to A$ $\quad \dfrac{L(W,1)}{\Omega} = \dfrac{\deg \phi}{Ni}$, $\quad \Omega = \pi i \displaystyle\int_{X_0(N)} \omega \wedge \bar{\omega}$

<u>Theorem</u> (Floch) $p \geq 5$, good red., $A[p] \circlearrowleft G_{\mathbb{Q}}$ image $GL_2(\mathbb{F}_p)$,
then ① $H^1_f(\mathbb{Q}, T \otimes \mathbb{Q}_p) = 0$
② $H^1_f(\mathbb{Q}, T \otimes \mathbb{Q}_p/\mathbb{Z}_p)$ Finite, annihilated by $\deg \phi$

<u>Wiles</u> Replace V by $\bar{V}$, $\bar{V}$ irred. modular

$$\begin{tikzcd} R \arrow[rr, "\sim"] \arrow[dr] & & \mathbb{T}_m \arrow[dl] \\ & \mathcal{O} & \end{tikzcd}$$

Corresponds to lattice in V

Says something about infinitesimal deformations; an extension would give map to $\mathcal{O}[\varepsilon]/\varepsilon^2$, but $\mathbb{T}_m$ is reduced in some generality.
Get close to an exact formula for ②

Weaker result like $R^{red} \xrightarrow{\sim} \mathbb{T}$

would be good enough for modularity but not enough to study Bloch-Kato.

An example

Let $\bar{\rho}: G_{\mathbb{Q}} \to GL_2(k)$ be absolutely irred., odd, unram at p

Let R be the minimal (Fix det.) deformation ring of $\bar{\rho}$

Unramified everywhere (after ramification in $\bar{\rho}$)

Then $R \cong \mathbb{T}_{\mathfrak{m}}$

$$\mathbb{T} \subseteq \mathrm{End}(\underbrace{H^1(X_1(N)}_{/\mathbb{Z}_p}, \omega^1))$$

IF $\bar{\rho}$ had image $GL_2(\mathbb{F}_7)$, $\Rightarrow$ no characteristic 0 modular lift (Deligne-Serre)

$\Rightarrow R$ torsion; (given $R \cong \mathbb{T}$) on the other hand if $\mathbb{T}$ is defined acting on a space of classical modular forms, no torsion, so $R = 0$

This is why we need H^1 instead of H^0 — leaves room for torsion (and by Serre duality in char 0 has same points as a Hecke algebra on classical modular forms)

Suppose you have such a result and want to use it to say something about Bloch-Kato

$\pi: R \simeq \mathbb{T} \to \mathcal{O}$

$\mathrm{Ker}\,\pi = I$

In case of Wiles,

$$\underbrace{\# I/I^2}_{\text{Selmer group}} = \underbrace{\# \mathcal{O}/\eta}_{\text{congruence module, related to adjoint L-values}} \quad (\eta = \pi(\mathrm{Ann}\, I))$$

Don't have exactly this in general

Can still deduce for wt 1 modular form, $H^1_F(\mathbb{Q}_S, Ad(V)) = 0$

V = Artin rep, modular, in char 0

Can translate as classgroup has no copy of $\mathbb{Z}_p$.

Can deduce also $H^1_F(\mathbb{Q}_S, T \otimes \mathbb{Q}_p/\mathbb{Z}_p)$ is finite, trivial for $p \gg 1$.

Let $X/\mathbb{Q}$ be a curve of genus $g=2$

Let $A = \mathrm{Jac}(X)$

Prop If A does not have CM, then ordinary primes p have density 1

Def A "good ordinary prime" ordinary,

$$\alpha\beta(\alpha^2-1)(\beta^2-1)(\alpha-\beta)(\alpha\beta-1) \neq 0 \bmod p$$

for α, β unit roots of Frob. — Also have density 1

Theorem Let $X/\mathbb{Q}$ be a "modular" curve of genus $g=2$,

let $T = T_p(A)$ $\quad V = T \otimes \mathbb{Q}_p$

$$V \otimes V = \Lambda^2 V \oplus \mathrm{Sym}^2 V$$
$$16 = 6 + 10$$
$$= 1 + 5 + \boxed{10} \text{ — "Ad V"}$$

Weil pairing

A not isog. $/\mathbb{Q}$ to a product of E.C.

① $H^1_F(\mathbb{Q}, \mathrm{Ad}\, T \otimes \mathbb{Q}_p) = 0$

For $p \gg 1$ good ordinary primes

② $H^1_F(\mathbb{Q}, \mathrm{Ad}\, T \otimes \mathbb{Q}_p/\mathbb{Z}_p) = 0$ under same hypothesis.

Modular: $L(H^1(X), s) = L(\pi, s)$

Siegel modular form

Idea: Prove a modularity lifting theorem for symplectic (4-dim'l rep.)

$$R_\infty \twoheadrightarrow \mathbb{T}_\infty$$

$\underline{\underline{R}}$ $\quad \mathbb{Z}_p[[x_1, \ldots, x_n]]/(f_1, \ldots, f_n)$

in general $\quad \mathbb{Z}_p[[x_1, \ldots, x_n]]/(f_1, \ldots, f_{n+\ell_0})$

On the automorphic side

wt. 1 mod p $\quad H^0(X_1(N)_{\mathbb{F}_p}, \omega)$ $\quad$ 0
$\quad H^1(X_1(N)_{\mathbb{F}_p}, \omega)$ $\quad \ell_0$

Hope to patch complex of 2 terms to show T_∞ is big.

1. Work with $H^*(X, \mathcal{E})$
   produce Galois rep's
2. The Galois rep.'s satisfy local-global compatibility
3. Vanishing of torsion classes after localizing at $\mathfrak{m}$ outside of range $(*, \ldots, * + \ell_0)$

- Issue of defining T_p

Theorem Let $r : G_{\mathbb{Q}} \to GSp_4(\overline{\mathbb{Q}}_p)$ be a rep. crystalline good ordinary with Hodge-Tate weights $(0, 0, k-1, k-1)$
$\bar{r}$ has big image and is modular (Katz/Siegel)
$p > k-1$
r minimally ramified
$\Rightarrow r$ modular if $k \geq 4$

If you tried to prove elliptic curves modular using this, would need $H^*(X, \omega^{\otimes 2}) = H^0(X, \mathcal{O}_X)$ to vanish
$= k$ – really only need Eisenstein, not vanishing.