Title: Cohomology of Arithmetic Groups

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Gelfand: Introduce Representation Theory

2 flavours

I. rep of $GL_2(\mathbb{Q}_p)$

$$\varinjlim H^1(X(p^n), \mathbb{C}) \quad \circlearrowleft GL_2(\mathbb{Z}_p)$$

$$GL_2(\mathbb{Z}/p^n\mathbb{Z}) \qquad \vdots$$

$$GL_2(\mathbb{Q}_p)$$

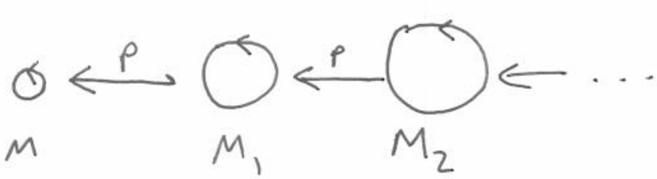
II. rep of $GL_2(\mathbb{R})$

$$\mathfrak{g}\ \mathfrak{gl}_2$$

<u>Towers</u>

<u>Ex.0.</u> $X(1) \leftarrow X(p) \leftarrow X(p^2) \leftarrow X(p^3) \leftarrow \cdots$

<u>Ex.1.</u> $M = S^1 \quad \Gamma = \pi_1(M) = \mathbb{Z}$



Ex 2. Let Σ_g = surface of genus $= g$

$$\gamma \in Isom(\Sigma_g, \Sigma_g)$$

$$M = \Sigma_g \times [0,1] \Big/ \Sigma_g \times \{0\} \xrightarrow{\sim} \Sigma_g \times \{1\}$$

$$M \longrightarrow S^1$$

$M =$ 

Ex 3. $S = \{p, \infty\}$ $\quad \mathbb{Q}_S$ = max. ext. of $\mathbb{Q}$ unramified at S

$$\Gamma = Gal(\mathbb{Q}_S / \mathbb{Q}) \qquad\qquad \Gamma = \pi_1(\mathbb{Z}[\tfrac{1}{p}])$$

$$\mathbb{Q}(\zeta_{p^n}) \subseteq \mathbb{Q}_S \qquad\qquad \Gamma_n = \pi_1(\mathbb{Z}[\tfrac{1}{p}, \zeta_{p^n}])$$

$$\begin{tikzcd}
 & & & \mathbb{Z}_p^\times \arrow[d, phantom, "\cong"] & \\
0 \arrow[r] & Gal(\mathbb{Q}_S / \mathbb{Q}(\zeta_{p^\infty})) \arrow[r] \arrow[d, equal] & \Gamma \arrow[r] & Gal(\mathbb{Q}(\zeta_{p^\infty})) \arrow[r] \arrow[d, phantom, "\cup"] & 0 \\
0 \arrow[r] & {} \arrow[r] & \Gamma_n \arrow[r] & 1 + p^n \mathbb{Z}_p \arrow[r] & 0
\end{tikzcd}$$

Two principles

(A) Often the entire tower is easier to understand than the individual layers.

(B) This is true even when there does not exist an "object" at the top of the tower

G = finite group

$Y \to X$ Galois group G

$C^\bullet_X$ cochain complex free $\mathbb{Z}$-modules

$C^\bullet_Y$ " " free $\mathbb{Z}[G]$-modules

Say X = compact curve (genus g)

$$\begin{array}{ccccc} \mathbb{Q} & - & \mathbb{Q}^{2g} & - & \mathbb{Q} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{Q}[G] & - & \mathbb{Q}[G]^{2g} & - & \mathbb{Q}[G] \end{array}$$

$\Rightarrow H^1(Y) = (2g-2)[\mathbb{Q}[G]] + 2[\mathbb{Q}]$

more generally $H^*(X, \mathbb{Q}) = H^*(Y, \mathbb{Q})^G$

What happens with coeff over $\mathbb{Z}$? A?

$Y \to X$ Galois gp $= G$

$$H_i(G, H_j(Y, A)) \Rightarrow H_{i+j}(X, A)$$

$A =$ field of char 0 $\quad (|G|^{-1} \in A)$

$$H_0(Y, A)_G = H_j(X, A)$$

$Y = S^3 \qquad X = S^3/G \qquad G$ (no fixed points)

$$\left.\begin{array}{l} H_*(Y, \mathbb{Z}) \\ H_*(X, \mathbb{Z}) \end{array}\right\} = \begin{cases} \mathbb{Z}, & i = 0, 3 \\ 0, & \text{else} \end{cases}$$

$H_3(Y, \mathbb{Z}) = \mathbb{Z}$

$$\begin{array}{cccccc}
\mathbb{Z} & H_1(G,\mathbb{Z}) & H_2(G,\mathbb{Z}) & H_3(G,\mathbb{Z}) & & \cdots \\
0 & 0 & 0 & 0 & & \\
0 & 0 & 0 & 0 & & \\
\boxed{\mathbb{Z}} & H_1(G,\mathbb{Z}) & H_2(G,\mathbb{Z}) & H_3(G,\mathbb{Z}) & H_4(G,\mathbb{Z}) & \cdots
\end{array}$$

$\Rightarrow H_1(G, \mathbb{Z}) = 0$ (G perfect)

$H_2(G, \mathbb{Z}) = 0$ (triv. Schur mult.)

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$$0 \leftarrow H_3(G,\mathbb{Z}) \leftarrow H_3(X,\mathbb{Z}) \leftarrow H_3(Y,\mathbb{Z}) \leftarrow H_4(G,\mathbb{Z}) \leftarrow 0$$

$$H_3(G,\mathbb{Z}) \cong \mathbb{Z}/N\mathbb{Z}, \quad H_3(X,\mathbb{Z}) = \mathbb{Z}, \quad H_3(Y,\mathbb{Z}) = \mathbb{Z}, \quad H_4(G,\mathbb{Z}) = 0$$

$N = \deg G$

$$H_*(G,\mathbb{Z}) = \begin{cases} \mathbb{Z}, & * = 0 \\ \mathbb{Z}/n\mathbb{Z}, & * = 4k+3 \\ 0, & \text{else} \end{cases}$$

$G = \tilde{A}_5 \qquad n = 120$

$G = 1 \qquad n = 1$

Ex. S^n has a quotient by $G \neq 1$ which is a hom. sphere $\iff n = 0, 1, 3$ (Swann)

$Y \to X \qquad$ Galois gp $= G$

$H_*(Y, A) \circlearrowleft G \qquad$ mod $A[G]$

$A = \mathbb{Z}_p \qquad G = \mathbb{Z}/p^k\mathbb{Z}$

$$H_*(Y, \mathbb{Z}_p) \circlearrowleft \frac{\mathbb{Z}_p[x]/(x^{p^k}-1)}{\mathbb{F}_p[x]/(x^{p^k}-1)}$$

$$\mathbb{F}_p[x]/(x^{p^k}-1) \cong \mathbb{F}_p[t]/t^{p^k}$$

$$\underbrace{M \leftarrow M_1 \leftarrow M_2 \leftarrow M_3 \leftarrow \cdots \quad M_n}_{\mathbb{Z}/p^n\mathbb{Z}}$$

$$H_*(M_n, \mathbb{Z}_p) \circlearrowleft \mathbb{Z}_p[\mathbb{Z}/p^n\mathbb{Z}]$$

$$\varprojlim H_*(M_n, \mathbb{Z}_p) \circlearrowleft \mathbb{Z}_p[[\mathbb{Z}_p]] \cong \mathbb{Z}_p[[T]]$$

$$T = [\gamma] - 1$$

M

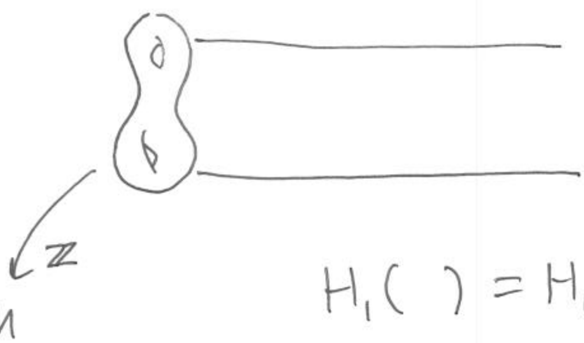
$$\varprojlim H_i(M_n, \mathbb{Z}_p) \circlearrowleft \mathbb{Z}_p[[T]]$$

$\mathbb{Z}_p^{2g} \curvearrowleft \gamma_*$ $\quad \mathbb{Z}_p[[T]]$

$1+T$

$A(T)$

$\Sigma_g \times \mathbb{R}$



$$H_1(\;\;) = H_1(\Sigma_g) = \mathbb{Z}^{2g}$$

$\mathbb{Z}[x, x^{-1}]$ $x \curvearrowright \gamma_*$

$A(x)$

$$X(1) \leftarrow X(p) \leftarrow X(p^2)$$

$$\mathbb{Z}[PSL_2 \mathbb{Z}]$$

$$\mathbb{Z}_p[[SL_2 \mathbb{Z}_p]]$$

$$\Gamma = SL_N \mathbb{Z}$$

$$\Gamma_N(p^k) = \text{cong sub}$$

$$\varinjlim H^d(\Gamma_N(p^k), \mathbb{F}_p) \circlearrowleft SL_N \mathbb{Q}_p$$

$d=1$ (or $d=2$) $N \gg 1$.