Title: Aspects of the Smooth Representation Theory of p-adic Groups

Speaker: Rachel Ollivier

Date: 2014.08.20 Time: 11:30am

$R=$ field of coeffs of the representation

$R=\mathbb{C}$ or $\overline{\mathbb{F}_p}$

$F=$ locally compact nonarch. field

residue char p $\quad q=p^f$

$\mathcal{O}$ ring of integers $\quad \mathcal{P}$ max ideal $\quad \mathcal{O}/\mathcal{P}=\mathbb{F}_q$

Ex: • $F=\mathbb{Q}_p$ or $F/\mathbb{Q}_p$ finite

$\mathcal{O}=\mathbb{Z}_p$

$\mathcal{P}=p\mathbb{Z}_p$

$\mathcal{O}/\mathcal{P}=\mathbb{F}_p$

$G=F$-points of a connected reductive group. Here $G=GL_n(F)$. Often $n=2$

- $F = \mathbb{F}_q((t))$ $\quad \mathcal{O}/\mathcal{P} = \mathbb{F}_q$

$\mathcal{O} = \mathbb{F}_q[[t]]$

$\mathcal{P} = t\mathbb{F}_q[[t]]$

A smooth rep of G is a rep V such that $\forall v \in V$ $\mathrm{Stab}_G(v)$ is open.

Example: $\mathbb{Q}_p$ = completion of $\mathbb{Q}$ for $|\cdot|_p = p^{-v_p(\cdot)}$

$\mathbb{Q}_p$ = locally compact totally disconnected

Fund. system of neighborhoods of 0.

$(p^i \mathbb{Z}_p)_{i \geq 0}$

$\mathbb{Z}_p = \{x : |x| \leq 1\}$ compact

$p\mathbb{Z}_p = \{x : |x| < 1\}$

$\mathbb{Q}_p^\times$ is locally a pro-p group

Fund system of neighborhoods of 1 $\quad U_i = 1 + p^i \mathbb{Z}_p$

$U_i / U_{i+1} = p$ group

$\mathbb{Z}_p^\times$ = unique max compact subgp of $\mathbb{Q}_p^\times$

$\mathbb{Z}_p^\times / U_1 = \mathbb{Z}/(p-1)\mathbb{Z}$

Smooth Characters

$\varphi : \mathbb{Q}_p^\times \longrightarrow \mathbb{C}^\times$

φ determined by $\varphi(p)$ and

$\varphi|_{\mathbb{Z}_p^\times}$ $\exists i \geq 1$ $\varphi|_{U_i}$ is kernel.

So $\varphi|_{\mathbb{Z}_p^\times}$ factors through $\mathbb{Z}_p^\times / 1+p^i\mathbb{Z}_p$

Smooth characters $\varphi : \mathbb{Q}_p^\times \longrightarrow \overline{\mathbb{F}}_p^\times$

φ det by $\varphi(p)$ and $\varphi|_{\mathbb{Z}_p^\times}$ $\qquad \mathbb{Z}_p^\times = \mathbb{Z}/(p-1)\mathbb{Z} \times 1+p\mathbb{Z}_p$

$\exists i \geq 1$ φ is trivial on $1+p^i\mathbb{Z}_p$

So $\varphi|_{1+p\mathbb{Z}_p}$ factors through $1+p\mathbb{Z}_p / 1+p^i\mathbb{Z}_p$

Lemma: A p-group acting on a $\overline{\mathbb{F}}_p$-vector space has a nonzero fixed vector.

So $\varphi|_{1+p\mathbb{Z}_p}$ trivial

Now $G = GL_n(F)$ $n \geq 1$ $\qquad K = GL_n(\mathcal{O})$ max open compact of G

$K_i = 1 + M_n(\mathcal{P}^i)$ for $i \geq 1$

$K/K_1 = GL_n(\mathbb{F}_q)$ $\qquad$ K_i/K_{i+1} is a p-group

$\mathbb{B} = \begin{pmatrix} * & * \\ & * \end{pmatrix}$ $\qquad$ $(K_i)_{i\geq 1}$ = fund. system of neighborhoods of 1.

$\triangledown$

$\mathbb{U} = \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}$

Let I be the preimage of $\mathbb{B}$ in K

Let I_1 be the preimage of $\mathbb{U}$ in K

I = Iwahori subgroup

I_1 = pro-p Iwahori subgroup

For $n=2$. The semisimple building of G is a tree

$G/KZ \simeq \underset{\substack{\circlearrowleft \\ G \text{ transitively}}}{\underline{\text{Vertices}}}$ = lattices of $F^2 = Fb_1 \oplus Fb_2$ up to rescaling

$x_0 = [\mathcal{O}b_1 \oplus \mathcal{O}b_2]$

$\mathrm{Stab}_G(x_0) = KZ$

Z is center of G.

y_1, y_2 2 vertices. They are neighbors if

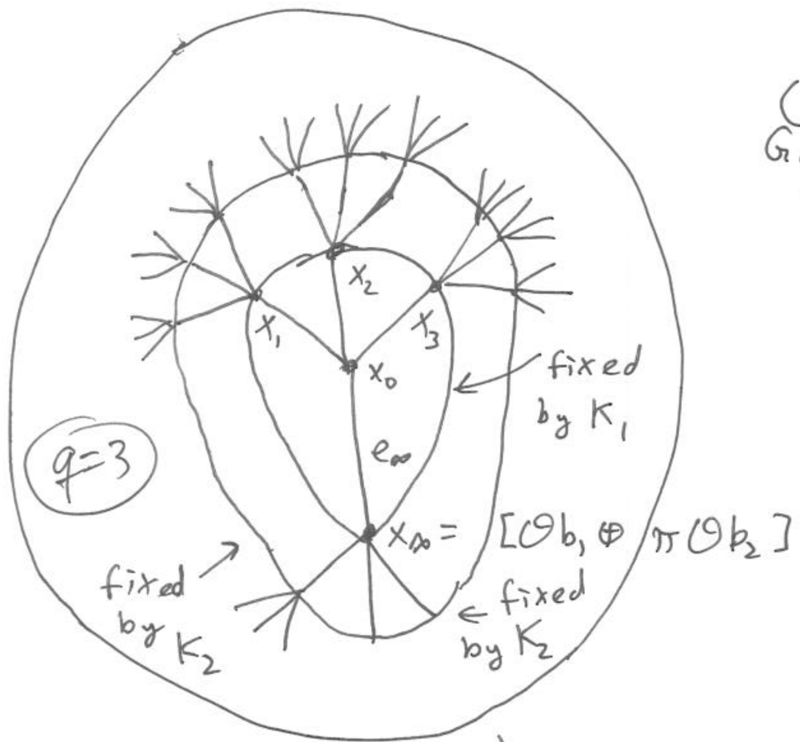
$\exists\, L_1, L_2 \quad y_1 = [L_1] \quad y_2 = [L_2]$

$\pi L_2 \subsetneq L_1 \subsetneq L_2$

$\{0\} \subsetneq \underbrace{\pi L_2 \backslash L_1}_{\text{line in } \mathbb{F}_q^2} \subsetneq \underbrace{\pi L_2 \backslash L_2}_{\mathbb{F}_q^2}$

π uniformizer of F

$\mathcal{P} = \pi\mathcal{O}$



$\downarrow$
$\mathbb{P}^1(\mathbb{Q}_p)$

Edges: Stabilizer of e_∞ is generated by I and $\begin{pmatrix} 0 & 1 \\ \pi & 0 \end{pmatrix}$

G acts transitively

$$\boxed{\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^i\mathbb{Z}}$$

The stabilizer of a point in $\mathbb{P}^1(\mathbb{Q}_p)$ is a Borel subgroup of $G = GL_2(F)$

$$B = \begin{pmatrix} * & * \\ & * \end{pmatrix} = \text{Stab of } \mathcal{O}_p b_1$$

Goal: Explore $\mathrm{Rep}_{\mathbb{C}}\, G$ $\quad \mathrm{Rep}_{\overline{\mathbb{F}}_p}\, G$

Homological properties?

Tools: • representation theory of finite groups $(GL_n(\mathbb{F}_q))$

- understand $V \in \mathrm{Rep}_R G$ via $\mathrm{Rep}_R GL_n(\mathbb{F}_q)$

  V^{Ω} = Ω-fixed vectors in V for various Ω

- "localization" of $V \in \mathrm{Rep}_{\mathbb{C}} G$ via coefficient systems on the building [Schneider-Stohler]

I. Representations of $G = GL_n(\mathbb{F}_q)$

i) let $\Omega < G$ $\qquad$ ex: $\Omega = \mathbb{B} = \begin{pmatrix} * & & * \\ & \ddots & \\ & & * \end{pmatrix}$

$\qquad = \mathbb{U} = \begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix}$

$$V^{\Omega} = \mathrm{Hom}_{\Omega}(1, V) = \mathrm{Hom}_{G}(\mathrm{ind}_{\Omega}^{G} 1, V)$$

$$\mathrm{ind}_{\Omega}^{G} 1 \cong R[G/\Omega]$$

$$\mathcal{H}_R(G, \Omega) = \mathrm{End}_G(R[G/\Omega]) = R[\Omega \backslash G / \Omega] \text{ with convolution}$$

Get a function: $\mathrm{Rep}_R G \longrightarrow \mathcal{H}_R(G, \Omega)$-module

$$V \longmapsto V^{\Omega}$$

<u>Example</u>: $\Omega = \mathbb{B}$ $\qquad \mathbb{B} \backslash G / \mathbb{B} = S_n$

$(S_n ; \{s_1, \dots, s_{n-1}\})$ is a Coxeter syst. (where $s_i = (i, i+1)$) with a length l

$\mathcal{H}_R(\mathbb{G}, \mathbb{B})$ has basis $(T_w = \mathrm{char}_{\mathbb{B} w \mathbb{B}})_{w \in \mathcal{S}_n}$

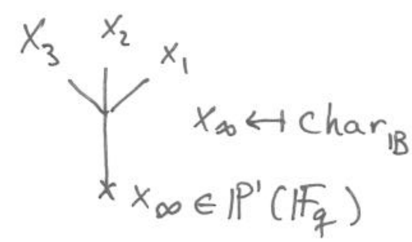
Relations: Braid: $T_w T_{w'} = T_{ww'}$ if $\ell(ww') = \ell(w) + \ell(w')$

$$\left(\begin{matrix} S_i \, S_{i+1} \, S_i = S_{i+1} \, S_i \, S_{i+1} \\ S_i = T_{s_i} \end{matrix} \right)$$

quadratic: $T_{s_i}^2 = (q\,1_R - 1)\, T_{s_i} + q \cdot 1_R$

Proof of the quadratic relation $\quad (n=2 \quad s := s_i)$

$R\left[{}_{\mathbb{B}}\backslash^{\mathbb{G}} \right] =$ functions on $\mathbb{P}^1(\mathbb{F}_q)$



$T_s^2(x_\infty)$?

$$T_s(x_\infty) = \mathrm{char}_{\mathbb{B} s \mathbb{B}} = \sum_{a \in \mathbb{F}_q} x_a$$

$$T_s^2(x_\infty) = \sum_{a \in \mathbb{F}_q} T_s(x_a) = \sum_{a \in \mathbb{F}_q} x_\infty + \sum_{\substack{b \in \mathbb{F}_q \\ b \neq a}} x$$

$$= q\, x_\infty + \sum_{b \neq \infty} x_b \Big(\sum_{a \in \mathbb{F}_q \setminus \{b\}} 1 \Big) = q\, x_\infty + (q-1)\, T_s(x_\infty)$$

Remarks: i) $R = \mathbb{C}$ then $\mathcal{H}_{\mathbb{C}}(\mathbb{G}, \mathbb{B})$ is a deformation of the case $q = 1$

$\mathbb{C}[\mathcal{S}_n]$ $\therefore$ $\mathcal{H}_{\mathbb{C}}(\mathbb{G}, \mathbb{B})$ semi simple.

2) $R = \overline{\mathbb{F}}_p$ $\quad T_{s_i}^2 = -T_{s_i}$

$n=2$ $\mathcal{H}_{\overline{\mathbb{F}}_p}(\mathbb{G}, \mathbb{B})$ is semisimple

$n \geq 3$ $\qquad$ not semisimple $\quad$ yet it is a Frobenius alg
$\Rightarrow$ has infinite global dimension

I. Representations of $GL_n(\mathbb{F}_q)$

2) $R = \mathbb{C}$ $\qquad \mathrm{Rep}_{\mathbb{C}} \mathbb{G}$ semisimple

$$\mathrm{Rep}_{\mathbb{C}} \mathbb{G} \supset \mathrm{Rep}_{\mathbb{C}}^{\mathbb{B}} \mathbb{G} = \text{rep } V \text{ generated by } V^{\mathbb{B}}$$

(also with $\mathbb{U}$ instead of $\mathbb{B}$)

<u>Theorem</u>: $\mathrm{Rep}_{\mathbb{C}}^{\mathbb{B}} \mathbb{G} \xrightarrow{\sim} \mathcal{H}_{\mathbb{C}}(\mathbb{G}, \mathbb{B})\text{-mod}$

$$V \longmapsto V^{\mathbb{B}}$$

same with $\mathbb{U}$ instead of $\mathbb{B}$

<u>Proof</u>: $V \mapsto V^{\mathbb{B}}$ has $\quad M \longmapsto M \otimes_{\mathcal{H}_R(\mathbb{G}, \mathbb{B})} R[\mathbb{B} \backslash \mathbb{G}]$

as a left adjoint

Prove that they are quasi-inverse

<u>key</u> $\quad W \in \mathrm{Rep}_{\mathbb{C}} \mathbb{G}$

$$P_{\mathbb{B}} : W \twoheadrightarrow W^{\mathbb{B}}$$

$$w \longmapsto \frac{1}{|\mathbb{B}|} \sum_{b \in \mathbb{B}} b.w.$$

Exercise: Classify the irr reps of $GL_2(\mathbb{F}_q)$ over $\mathbb{C}$

1) Classify the irr rep. in $\mathrm{Rep}^{\mathcal{U}}_{\mathbb{C}}\, \mathbb{G}$

Let V be such a rep: $\underset{G}{V}^{\mathcal{U}} \neq \{0\}$

Contained in $V^{\mathcal{U}}/\mathbb{B}$ Inflate $\chi : \mathbb{B} \to \mathbb{C}^{\times}$

$$0 \neq \mathrm{Hom}_{\mathbb{B}}(\chi, V|_{\mathbb{B}}) = \mathrm{Hom}_{\mathbb{G}}(\mathrm{ind}^{\mathbb{G}}_{\mathbb{B}}\, \chi, V)$$

So V is a quotient of a principal series representation.

ex: $\chi = 1$ $\quad \mathrm{ind}^{\mathbb{G}}_{\mathbb{B}}\, 1 = \mathbb{C}[\mathbb{B}\backslash\mathbb{G}]$

$$= \underbrace{\text{Constant}}_{\text{1-dim}} \oplus \underbrace{\Omega}_{q}$$

Semisimplify $\mathrm{ind}^{\mathbb{G}}_{\mathbb{B}}\, \chi$ for all χ

$$(q-1) + (q-1) + \frac{(q-1)(q-2)}{2}$$

There are $\frac{q(q-1)}{2}$ reps missing.

Obtain them by inducing certain indecomposable characters. of $\mathbb{F}_{q^2}^{\times} \subset GL_2(\mathbb{F}_q)$

Those irr rep. are called <u>cuspidal</u>

3) $R = \overline{\mathbb{F}_p}$ $\quad V \neq \{0\} \in Rep_{\overline{\mathbb{F}_p}} G$

$V^{U} \neq \{0\} \Longrightarrow$ V is a quotient of a principal series representation.