Clark Barwick: Redshift and higher categories

Today, we'll talk about the *redshift conjectures*. Morally, these say that invariants like K-theory increase chromatic complexity. We'll begin by trying to make this a little more precise.

Definition 1. A spectrum E is *of telescopic complexity* $\leq n$ *at the prime* p if, for any p-local finite spectrum V of type $\geq n$, $\pi_* V \wedge E \xrightarrow{\cong} \pi_* T \wedge E$ for $* \gg 0$, where $T = Tel(v_n)$, the unique mapping telescope of a v_n-self map

Note that the thick subcategory theorem implies that every spectrum *has* a telescopic complexity, and moreover that that telescopic complexity is *unique*.

Now, here is a conjecture.

Conjecture 1. *The iterated K-theory* $K^{(n)}(\mathbb{C}) = K(K(\cdots(K(\mathbb{C}))\cdots))$ *has telescopic complexity exactly* n *at every prime* p*.*

We immediately remark that we have the following theorem in this direction.

Theorem 1 (Suslin, Baas–Dundas–Richter–Rognes)**.** *This is true for* $n \leq 2$*.*

Note that $K(\mathbb{C})$ is just KU, or rather this is true after p-completion (and possibly taking connective covers?). Then, $K(ku)$ is a form of elliptic cohomology, i.e. it has telescopic complexity 2.

We believe that we currently have the tools to solve this conjecture. Today, we'll address the sub-question: What are the operations on $K^{(n)}(\mathbb{C})$? (This is all joint work with Saul Glasman.)

We first remark that we can consider $\mathbb{C}$ as an E_∞-ring spectrum (lol), and so these iterated K-theories are all sensible – they're E_∞-rings too.

We now begin with an algebraic digression. Let us attempt to understand the endomorphisms of the forgetful functor $U : \mathtt{Alg}_{\mathbb{C}} \to \mathtt{Set}$. Since this is representable, we find that $\mathrm{End}(U) \cong U(\mathbb{C}[x])$. This is an isomorphism of rings, first of all. But formal nonsense implies that $\mathbb{C}[x]$ must also admit a *co-*$\mathbb{C}$*-algebra* structure (in $\mathbb{C}$-algebras), and with this structure it represents the identity functor. But then, there's even more structure: if we have two morphisms we can *compose* them, and this yields a monoid structure on $\mathbb{C}[x]$, given by composition of functions: $(p, q) \mapsto p \circ q$. (Given $A \in \mathtt{Alg}_{\mathbb{C}}$, $p \in \mathbb{C}[x]$ acts on UA by $a \mapsto p(a)$.)

This all leads us to the following definition.

Definition 2 (Borger–Wieland, following Tall–Wraith)**.** A *plethory* over a ring R (or an R*-plethory*) is a co-R-algebra P in R-algebras, along with a monoid structure that enhances the functor $W_P : \mathtt{Alg}_R \to \mathtt{Alg}_R$ into a comonad.

These form a category, and it's actually not hard to see that $\mathbb{C}[x]$ is the *initial* $\mathbb{C}$-plethory.

There are other examples of plethories; we give some now.

Example 1. Define Λ to be the algebra of symmetric functions (say over $\mathbb{Z}$). Its comonad (i.e. composition) structure is essentially what's called the *Artin–Hesse map*. Inside of here we have the "p-typical" part $\Lambda(p)$.

We will see the relevance of these examples shortly. But we need one more observation. Note that $\mathbb{C}[x]$ acts simultaneously on all the $\mathbb{C}$-algebras, so it makes sense to ask for an *action* of a plethory. This motivates the following.

Definition 3. An *algebra* over a plethory P is a coalgebra over W_P.

Example 2. Since $\mathbb{C}[x]$ acts on all $\mathbb{C}$-algebras in a unique way, every $\mathbb{C}$-algebra admits a unique structure of an algebra over the plethory $\mathbb{C}[x]$.

We can now explain the notation.

Example 3. $W_\Lambda(A)$ is the ring of big Witt vectors on A; in this language, a Λ-algebra is precisely a λ-algebra! Moreover, $W_{\Lambda(p)}(A)$ is of course the ring of p-typical Witt vectors, and a $\Lambda(p)$-algebra is a "δ ring relative to p" (following Joyal's terminology) or a "θ_p-algebra" (following Bousfield).

Now, we can celebrate the fact that we have *new foundations* for algebraic topology. Nothing changes if we're sufficiently categorical; we can replace abelian groups with spectra, rings with E_∞-ring spectra, etc. In fact, we can even *categorify* this story, which is what we'll do now.

However, rather than try to construct sophisticated examples, we'll simply muse on the generalities.

Definition 4. We define $\mathtt{Perf}_{\mathbb{C}}$ to be the ($(\infty,1)$-)category of perfect $H\mathbb{C}$-modules – that is, of $H\mathbb{C}$-modules with finitely many homotopy groups, all of which are finite-dimensional.

Note that $\mathtt{Perf}_{\mathbb{C}}$ is also *symmetric monoidal*, i.e. it's a ring object in the $(\infty,2)$-category of $(\infty,1)$-categories. In fact, this lives in $\mathtt{Cat}^{rex}_{(\infty,1)}$, the symmetric monoidal $(\infty,2)$-category of idempotent-complete $(\infty,1)$-categories admitting all finite colimits (whose monoidal structure is the *categorical tensor product*, i.e. the object corepresenting functors that preserve colimits in each variable (so one should think of the finite colimits as the "addition")). And so $\mathtt{Perf}_{\mathbb{C}}$ is a *commutative algebra* in this $(\infty,2)$-category.

Now, let's run the same story as we did before. Namely, we look at the forgetful functor

$$U : \mathtt{CAlg}_{\mathtt{Perf}_{\mathbb{C}}} := \mathtt{CAlg}(\mathtt{Cat}^{rex}_{(\infty,1)})_{\mathtt{Perf}_{\mathbb{C}}/} \to \mathtt{Cat}_{(\infty,1)}.$$

So, what is $\mathrm{End}(U)$?

Example 4. Given any $\mathbb{C}$-variety, the perfect complexes of quasicoherent sheaves will form an object of the source.

Now, we begin by looking for the *free symmetric monoidal* $(\infty,1)$*-category* – but this is just Σ, the 1-groupoid of finite sets and automorphisms. Then, we want to understand the full subcategory $\mathtt{Perf}_{\mathbb{C}}[x] \subset \mathtt{Fun}(\Sigma, \mathtt{Perf}_{\mathbb{C}})$ spanned by those F such that $F(I) = 0$ for $|I| \gg 0$ (i.e., symmetric sequences that eventually peter out).

Since we're looking at *arbitrary* functors $\Sigma \to \mathtt{Perf}_{\mathbb{C}}$, this functor category has the *Day convolution* symmetric monoidal structure. To describe this, suppose we have $F, G : \Sigma \to \mathtt{Perf}_{\mathbb{C}}$. Then we define $F \otimes G$ to be the *left Kan extension* in the diagram

$$\begin{array}{ccc} \Sigma \times \Sigma & \xrightarrow{F \times G} & \mathtt{Perf}_{\mathbb{C}} \times \mathtt{Perf}_{\mathbb{C}} \\ \downarrow{\scriptstyle \otimes} & & \downarrow{\scriptstyle \otimes} \\ \Sigma & \xrightarrow[F \otimes G]{} & \mathtt{Perf}_{\mathbb{C}}. \end{array}$$

(If one really hated their audience, they could write $F \otimes G := \otimes_!(\otimes \circ (F \times G))$.) We can also describe this explicitly: $(F \otimes G)(I) \cong \mathrm{colim}_{I \cong J \sqcup K} F(J) \otimes G(K)$.

Here is a result; we don't know who to ascribe it to, but surely it's been known for a long time.

Theorem 2. $U(\mathtt{Perf}_{\mathbb{C}}[x]) \cong \mathrm{End}(U)$.

This is for the same reason we discussed: this corepresents the forgetful functor. (In fact, note that the composition on $\mathtt{Perf}_{\mathbb{C}}[x]$ is just the composition product of symmetric sequences!) From this, we get the following.

Corollary 1. $\mathtt{Perf}_{\mathbb{C}}[x]$ *is the initial plethory in* $\mathtt{CAlg}_{\mathtt{Perf}_{\mathbb{C}}}$.

Good! Let's take its K-theory. Here's a theorem, which looks trivial but actually takes a lot of work to prove.

Theorem 3. *The K-theory of a plethory is a plethory.*

(K-theory behaves nicely with respect to algebraic structures, but not with respect to coalgebraic structures a priori. Rather, this comes from its universal property.)

Thus, we obtain

$$\mathtt{CAlg}_{\mathtt{Perf}_{\mathbb{C}}} \xrightarrow{K} \mathtt{CAlg}_{K(\mathbb{C})},$$

and this takes plethories to plethories. Let's find out where $\mathtt{Perf}_{\mathbb{C}}[x]$ goes. Let's begin at level 0: we just have that $K_0(\mathtt{Perf}_{\mathbb{C}}[x]) = \Lambda$. (This might be ultimately due to MacDonald, in an appendix to his book.) Let's prove this. We know that

$$K_0(\mathtt{Perf}_{\mathbb{C}}[x]) \cong \mathrm{colim}_n \bigoplus_{i=1}^{n} K_0(\mathtt{Rep}_{\mathbb{C}}[\Sigma_i]) \cong \bigoplus_{n \geq 0} \mathtt{Rep}[\Sigma_n].$$

The punchline is the following: categorification takes a trivial plethory to a nontrivial plethory. This is just like redshift, which says that applying K-theory makes things more interesting!

Let's continue by observing that Λ admits Adams operations; passing to $\Lambda(p)$, we see that this contains a group that's dense in $\mathbb{Z}_p^\times$. Whereas we have $\mathtt{Perf}_{\mathbb{C}}[x]$ acting on $\mathtt{Perf}_{\mathbb{C}}$, then we get an action of $K(\mathtt{Perf}_{\mathbb{C}}[x])$ on $K(\mathbb{C})$, which reduces to the standard Adams operations action of Λ on $K_0(\mathbb{C})$. Completing at a prime p, we obtain something we'll call

$$K(\mathtt{Perf}_{\mathbb{C}}[x])_p^\wedge =: ku\lambda_p.$$

After p-completion, the K-theory of any symmetric monoidal ∞-category has an action of this plethory. Moreover, it's not hard to compute: $\pi_* ku\lambda_p = ku_* \otimes \Lambda_p^\wedge$.

In our last few minutes, we'll sketch how we go higher – not to $n = 2$, but to arbitrary n. To do this, we just need to replace 1 with n and 2 with $n+1$. So, we now consider $\mathtt{Cat}^{rex}_{(\infty,n)}$, the $(\infty, n+1)$-category of (∞, n)-categories which are *Morita-complete* (the higher-categorical analog of idempotent-complete) and admit all finite (∞, n)-colimits (in the lax sense). From this, we define $\mathtt{Perf}_{\mathbb{C}}^{(n)}$ inductively: $\mathtt{Perf}_{\mathbb{C}}^{(0)} = H\mathbb{C}$, and $\mathtt{Perf}_{\mathbb{C}}^{(n)}$ is symmetric monoidal $(\infty, 1)$-category of dualizable modules over $\mathtt{Perf}_{\mathbb{C}}^{(n-1)}$; that is,

$$\mathtt{Perf}_{\mathbb{C}}^{(n)} \in \mathtt{CAlg}(\mathtt{Cat}^{rex}_{(\infty,n)}).$$

Now, here's a fact: $K^{(n)}(\mathbb{C}) \simeq K(\mathtt{Perf}_{\mathbb{C}}^{(n-1)})$. (There's a notion of K-theory for which the right side makes sense and the equivalence is true.)

We can now attempt to classify all the operations here: the forgetful functor $U : \mathtt{CAlg}_{\mathtt{Perf}_{\mathbb{C}}^{(n-1)}/} \to \mathtt{Cat}_{(\infty,n)}$, and we can take $K(\mathrm{End}(U))$, and this will act canonically on the K-theory of any object of the source (in particular, its initial object).

This is the part where we'd like to have a punch-line – we'd like to be able to even compute K_0. But we can't do it, and we'd like to explain why. Down at $n = 1$, we had Maschke's theorem, which allows us to decompose the representations. But this fails at $n = 2$: we don't know that we have this for $\mathtt{Perf}_{\mathbb{C}}^{(2)}[x]$ (e.g. for "2-vector spaces"). Thus, we end with a question: Is there a useful analog of Maschke's theorem over $\mathtt{Perf}_{\mathbb{C}}$? We do know that $K(\mathtt{Perf}_{\mathbb{C}}^{(2)}[x])$ acts canonically on $K^{(2)}(\mathbb{C})$, which is supposed to be some form of elliptic cohomology. So maybe we can't compute all of K_0, but it's a plethory, and we can look for a submonoid that's dense in the Morava stabilizer group $\mathbb{S}_2$. For instance, Behrens–Lawson construct such a dense submonoid using isogenies of elliptic curves. It'd be really great to see that appearing here.

David Ben-Zvi asks an interesting related question; watch the video online to hear about it (something about Kac–Moody algebras and the failure of Maschke's theorem in certain contexts).