ASPECTS OF DIFFERENTIAL COHOMOLOGY

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Abstract. A differential cohomology theory is differential geometric refinement of a generalized cohomology theory (in the sense of algebraic topology). Examples naturally arise in physics or in the study of secondary invariants (e.g. Chern-Simons invariants). We discuss this notion from a higher categorical point of view. This leads to a natural decomposition of any differential cohomology theory which we illustrate with many examples. Moreover we show how to obtain a good integration theory and a notion of twisted differential cohomology and discuss some aspects and examples.

This was a chalk talk. The speaker decided not to share his hand-written lecture notes.

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1. Definitions and examples

A *cohomology theory* is a functor from smooth manifolds to graded abelian groups which satisfies Mayer-Vietoris, takes coproducts to products, and has $E^*(M \times I) \cong E^*(M)$ (i.e. it's homotopy invariant). We assume these are represented by spectra.

A *differential cohomology theory* is not assumed to be homotopy invariant, but retains the other properties above. It is represented by a sheaf of spectra $E : Mfd^{op} \to Spectra$ with $E^n(M) = \pi_{-n}(E(M))$.

Examples:

(1) Ordinary generalized cohomology theories (proven by Dugger).

(2) Let Ω be for differential forms. Then

$$(D_2\mathbb{R})^n(M) = \begin{cases} 0 & \text{if } n < 2 \\ \Omega^2_{df} M & \text{if } n = 2 \\ H^n(M, \mathbb{R}) & \text{otherwise} \end{cases}$$

is an example.

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(3) Let $\widehat{H}^2(M,\mathbb{Z})$ be iso classes of line bundles with connections. Then

$$(D_2\mathbb{Z})^n(M) = \begin{cases} H^{n-1}(M,\mathbb{R}/\mathbb{Z}) & \text{if } n < 2 \\ \widehat{H}^2(M,\mathbb{Z}) & \text{if } n = 2 \\ H^n(M,\mathbb{Z}) & \text{otherwise} \end{cases}$$

is an example. It's represented by Cheeger-Simons. Furthermore, the following commutes

$$\begin{array}{ccc} D_2\mathbb{Z} & \longrightarrow & D_2\mathbb{R} \\ \downarrow & & \downarrow \\ H^*(-,\mathbb{Z}) & \longrightarrow & H^*(-,\mathbb{R}) \end{array}$$

(4) $$(D_0K)^n(M) = \begin{cases} K^{-n+1}(M,\mathbb{R}/\mathbb{Z}) & \text{if } n < 0 \\ \widehat{K}^0(M) & \text{if } n = 0 \\ K^n(M) & \text{otherwise} \end{cases}$$

(5) Let A be a chain complex over $\mathbb{R}$. Let $h \in \mathbb{Z}$

$$D_nA = \begin{cases} 0 & \text{if } n < h \\ Z^h(\widetilde{\Omega}^*(M,A)) & \text{if } n = h \\ \widetilde{H}^h(M,A) & \text{otherwise} \end{cases}$$

(6) $E^n(M) = K_*^{alg}C^\infty(M,\mathbb{C})$. Note: this one does not satisfy Mayer-Vietoris. You can tweak it to do so by sheafifying. Note that $E^*(pt) = 0$ so this E is called *pure*.

Theorem 1.1 (Bunke-Nikolaus-Völkl)**.** *Given E, there is a universal homotopification $E \to hE$.*

When M is compact, we have a formula $(hE)(M) = \operatorname{colim}_\Delta E(M \times \Delta^\bullet)$

There is a universal pure $E \to pE$.

There is a pushout square

$$\begin{array}{ccc} E & \longrightarrow & hE \\ \downarrow & \searrow & \downarrow \\ pE & \longrightarrow & hpE \end{array}$$

The underlying ∞-category theory has been worked out by Urs Schreiber.

It's almost completely formal, but does not hold for $\mathbb{A}^1$-homotopy theory.

Examples

(1) Trivial

(2) $pD_2\mathbb{R} = D_2\mathbb{R}, hD_2\mathbb{R} \cong H^n_{dR}(-,\mathbb{R})$

(3) $pD_2\mathbb{Z} = D_2\mathbb{R}, hD_2\mathbb{Z} \cong H^n(-, \mathbb{R})$

(4) skip

(5) skip

(6) $hE = KU$, pE fits in a cofiber sequence $K^{alg}\mathbb{C} \to E \to pE$, and hpE sits in a cofiber sequence $K^{alg}\mathbb{C} \to ku \to hpE$

2. Refinements of differential cohomology theories

One can use the work of Hopkins-Singer, Freed to find refinements of ring spectra.

Let R be a spectrum, A a chain complex over $\mathbb{R}$, and an equivalence $c : R \wedge H\mathbb{R} \to HA$.

For example, if $R = KU, A = \mathbb{R}[\zeta, \zeta^{-1}]$ then this is useful.

Now let $\widehat{R}_h$ be a differential cohomology theory (so we have a family as h runs through $\mathbb{Z}$). Then take R to be $h\widehat{R}_h$ and take $p\widehat{R}_h = D_hA$. Now we simply need the maps $E \to hE$ and $E \to pE$ for this example. These maps can be obtained via

$$R^n(M) \to R^n(M, \mathbb{R}) \xrightarrow{c} H^n(M, A) \cong H^n_{dR}(M, A) = (hD_hA)^n(M)$$

Remark: of the family above we only care about one homotopy group. The rest of the theories are just there to ensure the Mayer-Vietoris condition.

Defn: $\widehat{R^n}(M) = \widehat{R}^n_n(M)$ is the refinement of the differential cohomology theory R.

Gepner-Bunke give functorial input

$$\begin{array}{ccc} R & \longrightarrow & R' \\ \downarrow & & \downarrow \\ HA & \longrightarrow & HA' \end{array}$$

where the vertical maps are given by c and A should be thought of as $\pi_*(R) \otimes \mathbb{R}$. This is used to get $\widehat{R} \to \widehat{R}'$.

For example, the map $MSpin \to KO$ gives a refinement $\widehat{MSpin} \to \widehat{KO}$.

We are now prepared to do integration theory. Let $H \in \widehat{H}^3(M, \mathbb{Z})$ and $f : \Sigma \to M$. Then $\int_c$ gives the map $f^*H \in \widehat{H}^3(\Sigma) \to \widehat{H}^n(pt) = \mathbb{R}/\mathbb{Z}$.

Many authors have worked this out for ordinary cohomology theories and K-theory. Bunke did it in complete generality.

3. Twisting

If R is E_∞ then a *twist* on M is a 1-1 assignment of any nice, invertible sheaf of R-modules over M to a map $M \to Pic_R$. Call it τ.

Given τ we can define $R^\tau(M) = \pi_0(\tau(M))$.

Defn: On the data (R, A, c), a *differential* twist over M is
- a topological twist τ
- a flat invertible sheaf $\mathcal{M}$ of DG-modules over $\Omega^*(-, A)$
- An equivalence $b : \tau \wedge H\mathbb{R} \xrightarrow{\sim} H\mathcal{M}$ called the twisted deRham isomorphism.

Think of this equivalence as $R^\tau(M) \otimes \mathbb{R} \to H^0_{dR}(\mathcal{M}(M))$

Proposition 3.1. *There is a good theory of integration in this setting.*

Examples of producing twisted deRham isomorphisms from actual geometric data.

(1) Let $\mathcal{A}$ be a system of A. Let $\mathcal{M} = \Omega^0(-, \mathcal{A})$. Let $R = KU, A = \mathbb{R}\langle\zeta, \zeta^{-1}\rangle$, and L a flat line bundle. Then $\mathcal{M} = SC(-, L)$ where SC is for super-connections.

(2) Let $w \in Z^1(\Omega^n(M, A))$

Let $R = KU, H \in \Omega^3(M), w = Hb$.

To find the equivalence b for the last example we need to classify all choices:

Theorem 3.2 (Bunke-Nikolaus)**.** *Let R be a nice spectrum (e.g. KU, KO, TMF, ku, ko, MSO, MSpin,...). Every topological twist admits a refinement to a differential (with $\mathcal{M} = \mathcal{M}(L, w)$). Furthermore, the refinement is unique (but non-canonical).*

The proof is basically brute force. You must use the twisted Atiyah-Hirzebruch spectral sequence, prove only one differential matters, and then compute.

Corollary: $\widehat{R}^{\tau, \mathcal{M}, \mathcal{A}}(M)$ only depends on τ.

Example:

When $R = KU, [b] \in H^3(M)$. Then you have $M \to K(\mathbb{Z}, 3) \to Pic_{KU}$ taking M-graded gerbes to topological twists on M. Let E_g denote the image of g.

Similarly, gerbes with connection are taken to differential twists on M. We denote this passage $(g, B, A) \to (E_g, \Omega^n(M)[\zeta, \zeta^{-1}], \sim)$.

Note: The connections on a given gerbe are a torsor over some $\mathbb{Z}/2$-lattice. One can use this to compute $\widehat{\mathcal{G}}$.

You can also investigate integration with respect to other orientations, e.g. given by maps $K(\mathbb{Z}, 2) \to KU$, $K(\mathbb{Z}, 3) \to tmf$, $K(\mathbb{Z}, n+1) \to E(n)$.

Hopefully these foundations will be useful for Chern-Weil computations.