Semiclassical measures of quantum cat maps

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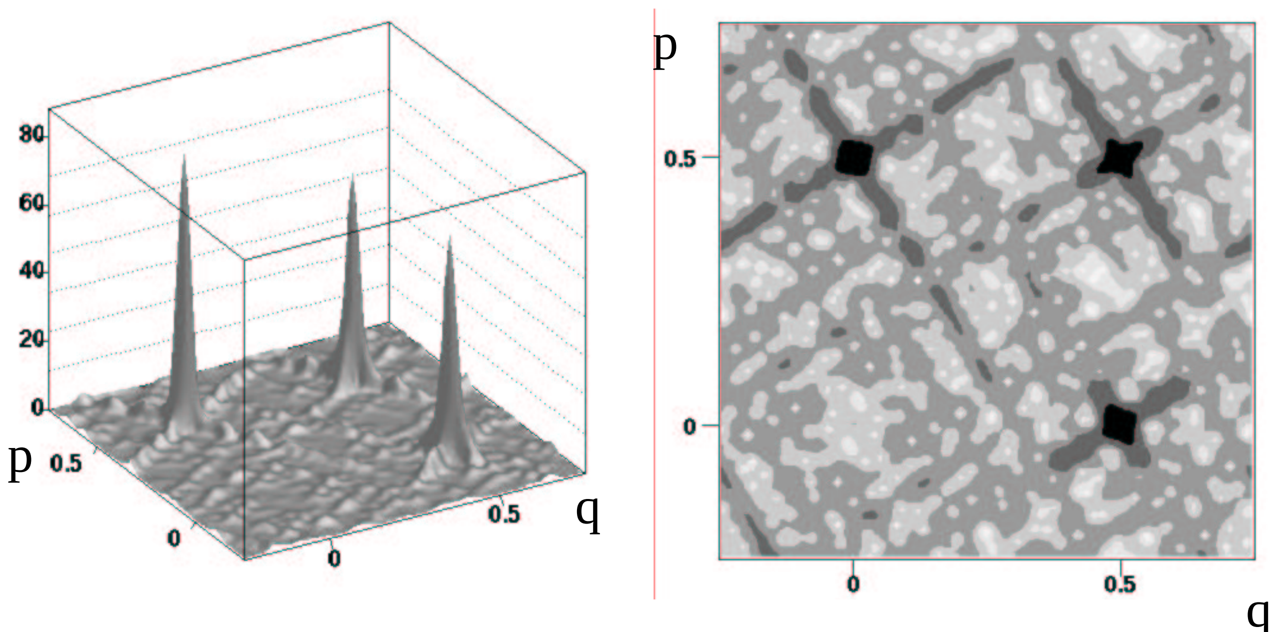
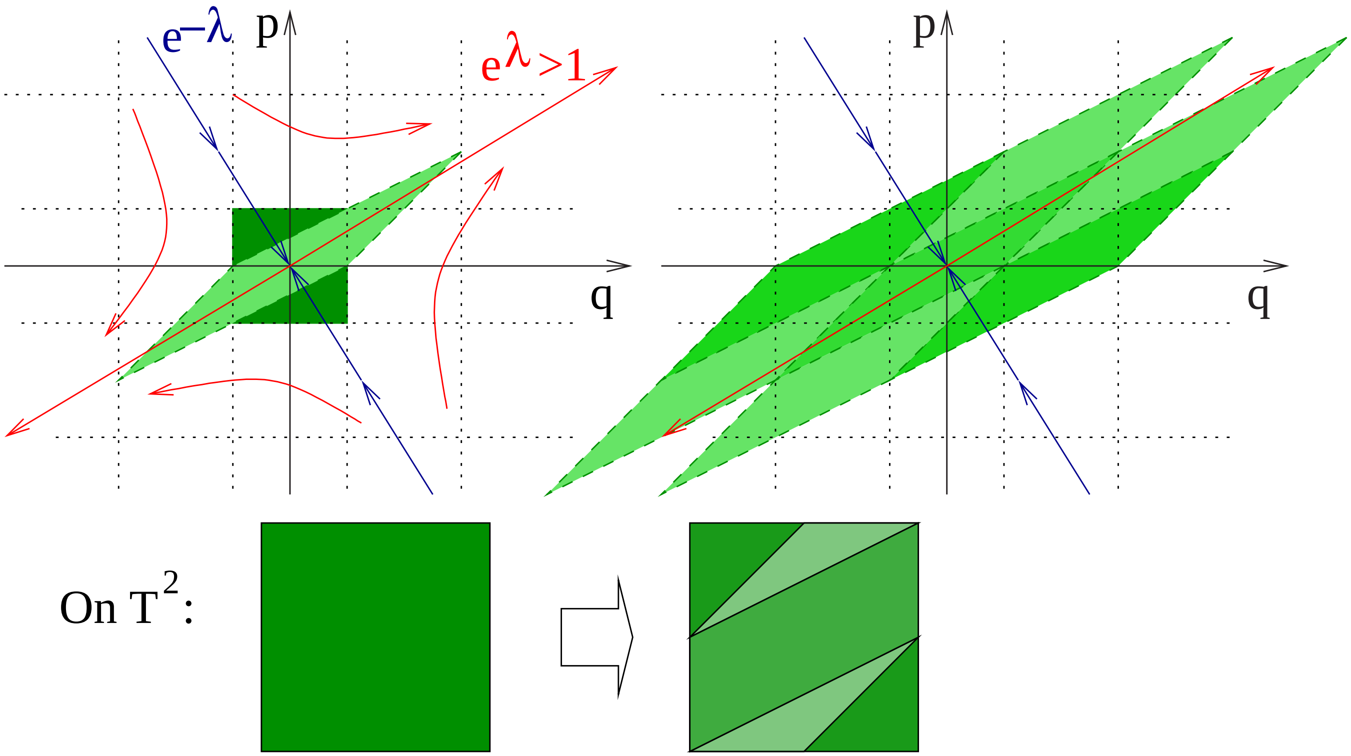


(a) (b)

Hyperbolic torus automorphisms (Arnold's cat maps)

We consider the map on the 2-dimensional torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, given by a **hyperbolic** matrix $M \in \mathrm{SL}(2,\mathbb{Z})$



$\lambda > 0$ uniform Lyapunov $\Rightarrow$ the map M is Anosov $\Rightarrow$ ergodic, mixing etc..

Many invariant measures $\mu \in \mathfrak{M}$: Lebesgue measure dx; periodic orbits $\delta_{\mathcal{P}}$ (rational coordin.). $\{\delta_{\mathcal{P}}\}$ are **dense** in $\mathfrak{M}$ [Sigmund].

Quantization of M [Han-Ber,DeEsp,Bou-DeBiè]

For any $\hbar > 0$, the linear map M on $\mathbb{R}^2$ is quantized into a metaplectic transformation $\hat{M}_\hbar$ unitary on $L^2(\mathbb{R})$.

$\forall v \in \mathbb{R}^2 \longrightarrow$ the quantum translation $\hat{T}_{v,\hbar} = \exp\{\mathrm{i}(\hat{q}v_2 - \hat{p}v_1)/\hbar\}$ acts on $\mathcal{S}'(\mathbb{R})$.

If $(2\pi\hbar)^{-1} = N \in \mathbb{N}$, the "space of torus states"

$$\mathcal{H}_N = \left\{ |\psi\rangle \in \mathcal{S}'(\mathbb{R}),\ \hat{T}_{(0,1),\hbar}|\psi\rangle = \hat{T}_{(1,0),\hbar}|\psi\rangle = |\psi\rangle \right\}$$

is nontrivial, and **invariant through** $\hat{M}_\hbar$ (if $M \in \Gamma_\theta$).

$\mathcal{H}_N =$the range of the projector $\hat{P}_{\mathbb{T}^2}$: $\mathcal{S}(\mathbb{R}) \to \mathcal{S}'(\mathbb{R})$

$$\hat{P}_{\mathbb{T}^2} = \hat{P}_{\mathbb{T}^2,\hbar} = \sum_{n \in \mathbb{Z}^2} (-1)^{N n_1 n_2}\, \hat{T}_{n,\hbar}.$$

$\mathcal{H}_N \approx \mathbb{C}^N$ can be given a Hilbert structure $\longrightarrow$ $\hat{M}_\hbar = \hat{M}_N$ is a $N \times N$ unitary matrix on $\mathcal{H}_N$: a "quantum map" on $\mathbb{T}^2$.

Semiclassical measures of M

We want to describe sequences of eigenstates $\{|\psi_{j,\hbar}\rangle\}_{\hbar\to 0}$ of $\hat{M}_\hbar$.

To each state $|\psi_\hbar\rangle \in \mathcal{H}_N$ is associated a Husimi measure $\rho_{\psi_\hbar}$.

We are interested in the weak$-*$ limits $\mu = \lim_{\hbar\to 0} \rho_{\psi_{j,\hbar}}$ for sequences of eigenstates. Any such limit $\mu \in \mathfrak{M}_{sc}$ is called a **semiclassical measure of** M.

Proposition. *[Egorov]* $\mathfrak{M}_{sc} \subset \mathfrak{M}$.

For an ergodic system (symplectic map/Hamiltonian flow), one has a general result: **Quantum Ergodicity**

Theorem. *[Schn, CdV, Zel, He-Ma-Ro, Ge-Le, Ze-Zw etc.]*

Let $\mathcal{M}$ be an ergodic map on $\mathbb{T}^2$, and $\hat{\mathcal{M}}_\hbar$ its quantization. For almost all sequences of eigenstates $\{|\psi_{j,\hbar}\rangle\}_{\hbar\to 0}$ of $\hat{\mathcal{M}}_\hbar$, the associated Husimi measures converge to the Lebesgue measure on $\mathbb{T}^2$.

This theorem holds in particular for the quantum cat map $\hat{M}_\hbar$.

Question: can some exceptional sequence of eigenstates converge towards another invariant measure?

Quantum unique ergodicity

Quantum unique ergodicity means that **all** semiclassical sequences of eigenstates converge to the Lebesgue measure: $\mathfrak{M}_{sc} = \{dx\}$.

QUE holds if $\mathcal{M}$ is a uniquely ergodic map: $\mathfrak{M} = \{dx\}$ [Mar-Rud].

QUE was recently proven [Lindenstrauss] for Hecke eigenstates of the Laplacian on arithmetic surfaces (all eigenstates?).

A counterexample to QUE was obtained by [Schubert *et al.*] by quantizing some ergodic (non-mixing) interval-exchange maps lifted on the torus.

For the **cat map** M, QUE was proven

- for "Hecke eigenstates" [Kurl-Rud]
- for all eigenstates along subsequences $\{\hbar_k\}$ [DeEs-Gra-Is,Ku-Ru].

These results use "hard" number theory.

Exceptional sequences exist for cat maps

Theorem 1. *[F-N-DB]*

For any periodic orbit $\mathcal{P}$ of M, there is a semiclassical sequence of eigenstates $\left\{|\Phi_{\hbar_k}\rangle\right\}_{\hbar_k \to 0}$ of $\hat{M}_{\hbar_k}$ whose Husimi densities weakly converge to $\frac{1}{2}dx + \frac{1}{2}\delta_{\mathcal{P}}$ as $\hbar_k \to 0$.

Since $\mathfrak{M}_{sc}$ is a closed subset of $\mathfrak{M}$, one gets:

Corollary. *For any $\mu \in \mathfrak{M}$, the inv. measure measure $\frac{1}{2}(dx + \mu) \in \mathfrak{M}_{sc}$.*

On the other hand, not all invariant measures can be semiclassical measures:

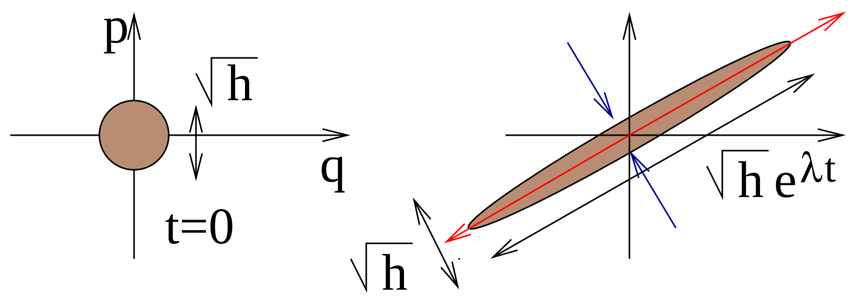
Theorem 2. *[F-N]*

If $\mu \in \mathfrak{M}_{sc}$, then its pure point and Lebesgue components satisfy $\mu_{pp}(\mathbb{T}^2) \leq \mu_{Leb}(\mathbb{T}^2)$, which implies $\mu_{pp}(\mathbb{T}^2) \leq 1/2$.

Main tool: time evolution of localized states.

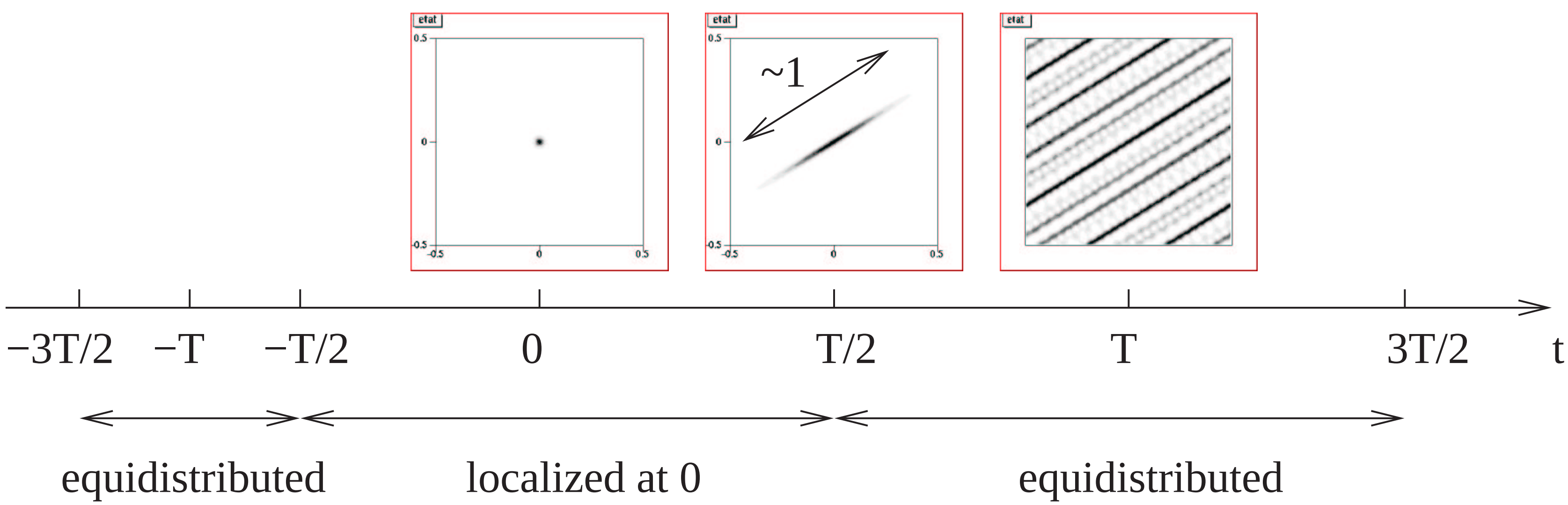
Time evolution of a coherent state

Take a coherent state ("circular" Gaussian wave packet) at the origin (fixed point) $|0_\hbar\rangle_{\mathbb{T}^2} = \hat{P}_{\mathbb{T}^2}|0_\hbar\rangle$. Study $|\psi(t)\rangle_{\mathbb{T}^2} = \hat{M}_\hbar^t|0_\hbar\rangle_{\mathbb{T}^2}$



The evolved state wraps itself on the torus $\Longrightarrow$ need to consider the Ehrenfest time

$$T_E(\hbar) = \frac{|\log 2\pi\hbar|}{\lambda}$$



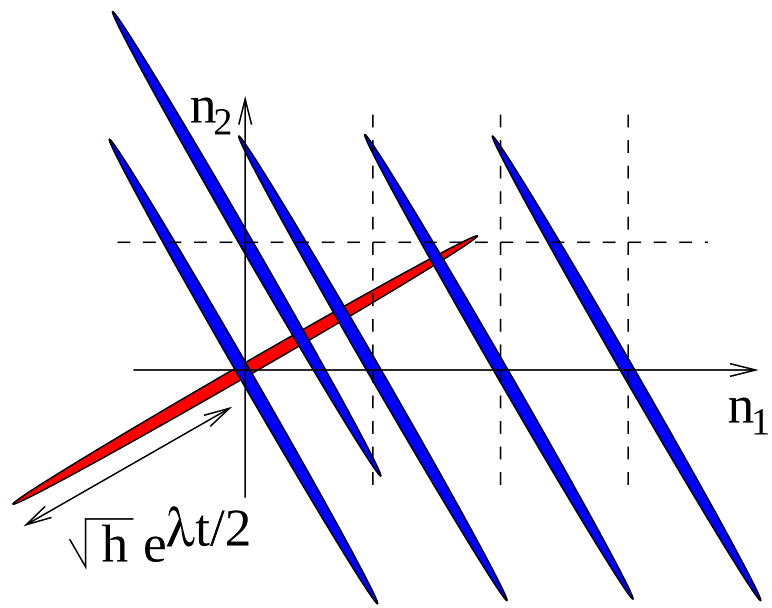
To test the spreading of $|\psi(t)\rangle_{\mathbb{T}^2}$, use the **autocorrelation function**

$$C(t) \stackrel{\text{def}}{=} {}_{\mathbb{T}^2}\langle 0_\hbar|\psi(t)\rangle_{\mathbb{T}^2} = {}_{\mathbb{T}^2}\langle\psi(-t/2)|\psi(t/2)\rangle_{\mathbb{T}^2}$$
$$= \sum_{n\in\mathbb{Z}^2} \mathrm{e}^{\mathrm{i}\delta_n}\langle\psi(-t/2)|\hat{T}_n|\psi(t/2)\rangle$$

For $t < T_E$, only the $n = 0$ term $\Longrightarrow C(t) \sim \mathrm{e}^{-\lambda t/2}$.

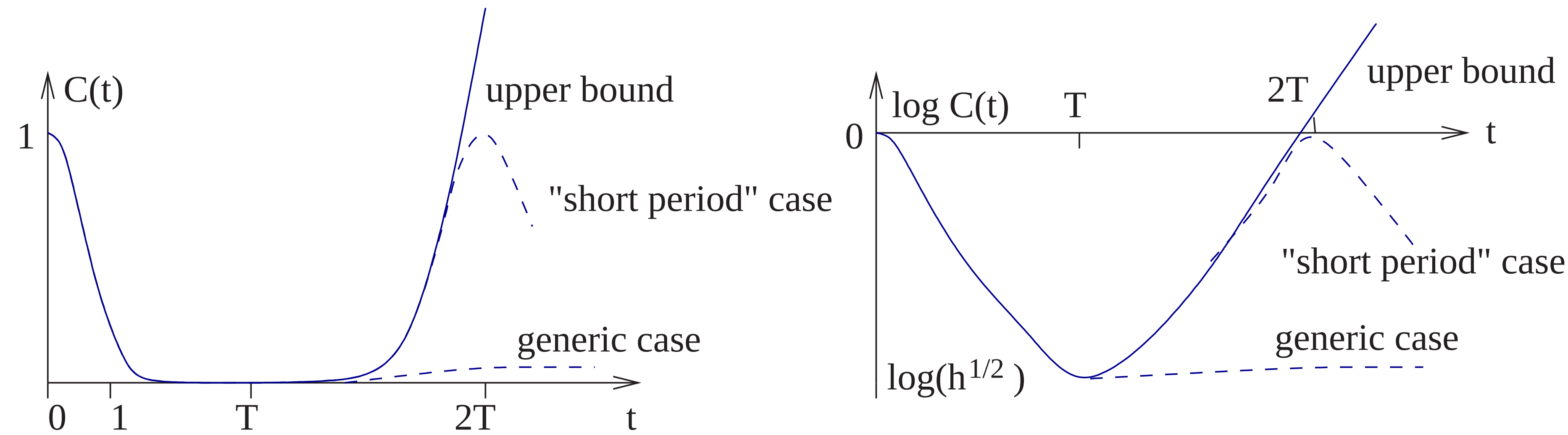
For $t > T_E$, contributions of $\mathcal{N}_t \sim \hbar\mathrm{e}^{\lambda t}$ homoclinic intersections. Each contribution $\simeq \mathrm{e}^{\mathrm{i}\varphi_n}\mathrm{e}^{-\lambda t/2}$
$\Longrightarrow C(t) \sim \mathrm{e}^{-\lambda t/2}\sum_1^{\mathcal{N}_t}\mathrm{e}^{\mathrm{i}\varphi_n}$.

If random phases,
$C(t) \sim \mathrm{e}^{-\lambda t/2}\sqrt{\mathcal{N}_t} \approx \sqrt{\hbar}$.



If rigid phases, $|C(t)| \sim \mathrm{e}^{-\lambda t/2}\mathcal{N}_t \simeq \hbar\mathrm{e}^{\lambda t/2}$

$|t| > 2T_E \;\rightarrow$ full revival of $|0_\hbar\rangle_{\mathbb{T}^2}$ possible.



Sketch of the autocorrelation function $C(t)$ (linear + logarithmic plots)

Transition localized $\rightarrow$ equidistributed

Consider a finite finite set of periodic orbits $\mathcal{S} = \{\mathcal{P}_1, \ldots, \mathcal{P}_s\}$, and an invariant probability measure $\delta_{\mathcal{S},\alpha} = \sum_i \alpha_i \delta_{\mathcal{P}_i}$.

Proposition 1. *[Bonechi-DeBièvre] Let a sequence of states* $\{|\psi_{\mathcal{S},\hbar}\rangle\}$ *converge to the measure* $\delta_{\mathcal{S},\alpha}$. *Then, the sequence* $\{|\psi'_{\mathcal{S},\hbar}\rangle \stackrel{\text{def}}{=} \hat{M}_\hbar^{T_E} |\psi_{\mathcal{S},\hbar}\rangle\}$ *converges to the Lebesgue measure.*

Notice: For any $\mathbf{k} \in \mathbb{Z}^2$, the plane wave $F_{\mathbf{k}}(q,p) = \exp\{2\mathrm{i}\pi(qk_2 - pk_1)\}$ is Weyl-quantized on $\mathcal{H}_N$ into the infinitesimal translation $\hat{T}_{h\mathbf{k}}$.

Therefore, equidistribution of $\{|\psi'_{\mathcal{S},\hbar}\rangle\}$ means that

$$\forall \mathbf{k} \in \mathbb{Z}^2, \quad \langle \psi'_{\mathcal{S},\hbar} | \hat{T}_{h\mathbf{k}} | \psi'_{\mathcal{S},\hbar} \rangle \xrightarrow{h\to 0} 0$$

Proposition 2. *Let* ν *be an invariant measure s.t.* $\nu(\mathcal{S}) = 0$, *and* $\{|\psi_{\nu,\hbar}\rangle\}$ *another sequence converging towards* ν.
Then, $\forall \mathbf{k} \in \mathbb{Z}^2$, $\langle \psi_{\nu,\hbar} | \hat{T}_{h\mathbf{k}} | \psi_{\mathcal{S},\hbar} \rangle \xrightarrow{h\to 0} 0$ *(obvious)*
and also $\langle \psi_{\nu,\hbar} | \hat{M}_\hbar^{-T_E} \hat{T}_{h\mathbf{k}} \hat{M}_\hbar^{T_E} | \psi_{\mathcal{S},\hbar} \rangle \xrightarrow{h\to 0} 0$ *(less obvious).*

Proof of Proposition 1

Exact Egorov property:

$$\boxed{\forall t \in \mathbb{Z}, \quad \hat{M}_\hbar^{-t} \hat{T}_{h\mathbf{k}} \hat{M}_\hbar^{t} = \hat{T}_{hM^{-t}\mathbf{k}}}$$

Therefore, for any $\mathbf{k} \in \mathbb{Z}^2 \setminus 0$,

$$\langle \psi'_{\mathcal{S},\hbar} | \hat{T}_{h\mathbf{k}} | \psi'_{\mathcal{S},\hbar} \rangle = \langle \psi_{\mathcal{S},\hbar} | \hat{T}_{hM^{-T_E}\mathbf{k}} | \psi_{\mathcal{S},\hbar} \rangle$$

The operator on the RHS is now a finite translation:

$$hM^{-T_E}\mathbf{k} = hM^{-T_E}\mathbf{k}^{stable} + hM^{-T_E}\mathbf{k}^{unstable} = \mathbf{k}^{stable} + \mathcal{O}(\hbar^2).$$

$\mathcal{S}$ is a set of *rational* points, and $\mathbf{k}^{stable}$ has an *irrational* slope
$\Longrightarrow \mathcal{S} + \mathbf{k}^{stable}$ is at a finite distance from $\mathcal{S}$.
$\Longrightarrow |\psi_{\mathcal{S},\hbar}\rangle$ and its translate by $\mathbf{k}^{stable}$ do not interfere.
$\square$

1st application: upper bound for scarring

Proof of Thm 2

Let $\{|\psi_\hbar\rangle\}$ be a sequence of eigenstates of $\hat{M}_\hbar$ converging towards μ.

Assume $\mu = \beta\delta_{\mathcal{S},\alpha} + (1-\beta)\nu$ with $\nu(\mathcal{S}) = 0$, and $0 \le \beta \le 1$.

Using a smooth function $\vartheta_\epsilon(x)$ localized near $\mathcal{S}$, we can construct a "microlocal projector" $\hat{\theta}_{\epsilon(\hbar)}$ such that

– $|\psi_{\mathcal{S},\hbar}\rangle \stackrel{\text{def}}{=} \hat{\theta}_{\epsilon(\hbar)}|\psi_\hbar\rangle$ converges to the measure $\beta\delta_{\mathcal{S},\alpha}$.

– $|\psi_{\nu,\hbar}\rangle \stackrel{\text{def}}{=} (1-\hat{\theta}_{\epsilon(\hbar)}|\psi_\hbar\rangle\}$ converges to the measure $(1-\beta)\nu$.

Now, we play with time evolution:

$$\langle\psi_\hbar|\hat{T}_{h\mathbf{k}}|\psi_\hbar\rangle = \langle\psi_\hbar|\hat{M}_\hbar^{-T_E}\hat{T}_{h\mathbf{k}}\hat{M}_\hbar^{T_E}|\psi_\hbar\rangle$$

$$\langle\psi_{\mathcal{S},\hbar}|\hat{T}_{h\mathbf{k}}|\psi_{\mathcal{S},\hbar}\rangle + \langle\psi_{\nu,\hbar}|\hat{T}_{h\mathbf{k}}|\psi_{\nu,\hbar}\rangle + c.t. = \langle\psi'_{\mathcal{S},\hbar}|\hat{T}_{h\mathbf{k}}|\psi'_{\mathcal{S},\hbar}\rangle + \langle\psi'_{\nu,\hbar}|\hat{T}_{h\mathbf{k}}|\psi'_{\nu,\hbar}\rangle + c.t.$$

as $\hbar \to 0$, $\quad \beta\delta_{\mathcal{S},\alpha}(F_\mathbf{k}) + (1-\beta)\nu(F_\mathbf{k}) = \beta dx(F_\mathbf{k}) + (1-\beta)\nu^?(F_\mathbf{k})$

The Lebesgue component on the LHS is in $\nu \Longrightarrow (1-\beta) \ge \beta$. $\square$

2d application: Quasimodes of maximal scarring

We want to construct a quasimode for $\hat{M}_\hbar$ by evolving the coherent state $|0_\hbar\rangle_{\mathbb{T}^2}$. For any $\phi \in [-\pi, \pi]$, we define:

$$|\Phi_\phi\rangle \stackrel{\text{def}}{=} \sum_{t=-T_E/2}^{3T_E/2} \mathrm{e}^{-\mathrm{i}\phi t}\, \hat{M}_\hbar{}^t |0_\hbar\rangle_{\mathbb{T}^2}.$$

This state can be split into $|\Phi_{\phi,loc}\rangle + |\Phi_{\phi,equi}\rangle$, with

$$|\Phi_{\phi,loc}\rangle \stackrel{\text{def}}{=} \sum_{t=-T_E/2}^{T_E/2} \mathrm{e}^{-\mathrm{i}\phi t}\, \hat{M}_\hbar{}^t |0_\hbar\rangle_{\mathbb{T}^2} \quad \text{and} \quad |\Phi_{\phi,equi}\rangle \stackrel{\text{def}}{=} \hat{M}_\hbar^{T_E} |\Phi_{\phi,loc}\rangle.$$

$|\Phi_{\phi,loc}\rangle$ is made of localized coherent states $\Longrightarrow$ is localized at 0.

From the simplicity of the autocorrelation function $C(t)$ for $|t| \leq T_E$, we can compute its norm $\|\Phi_{\phi,loc}\|_{\mathcal{H}_N} \sim S(\phi)\sqrt{T_E}$.

We have a sequence $\{|\Phi_{\phi,loc}\rangle_n\}$ converging to the measure δ_0.

- Prop. 1$\Longrightarrow$ $|\Phi_{\phi,equi}\rangle_n$ is equidistributed
- $\Longrightarrow$ $|\Phi_{\phi,loc}\rangle_n$ and $|\Phi_{\phi,equi}\rangle_n$ are "independent".

Consequences:

- $\| |\Phi_\phi\rangle \| \sim S(\phi)\sqrt{2T_E} \Longrightarrow |\Phi_\phi\rangle_n$ is a **quasimode** of $\hat{M}_\hbar$:

  $\|(\hat{M}_\hbar - \mathrm{e}^{\mathrm{i}\phi})|\Phi_{\phi,equi}\rangle_n\| \leq \frac{C}{\sqrt{T_E}}$

- the states $\{|\Phi_\phi\rangle_n\}$ converge to the semiclassical measure $\frac{\delta_0+dx}{2}$.

Similarly, by propagating a coherent state $|x_{0,\hbar}\rangle_{\mathbb{T}^2}$ localized at a point x_0 on a periodic orbit $\mathcal{P}$, one constructs quasimodes converging to the measure $\frac{\delta_{\mathcal{P}}+dx}{2}$.
These quasimodes are not yet eigenstates...

Figure 1: The two components of the quasimode at the origin for N=500, $\phi = 0$

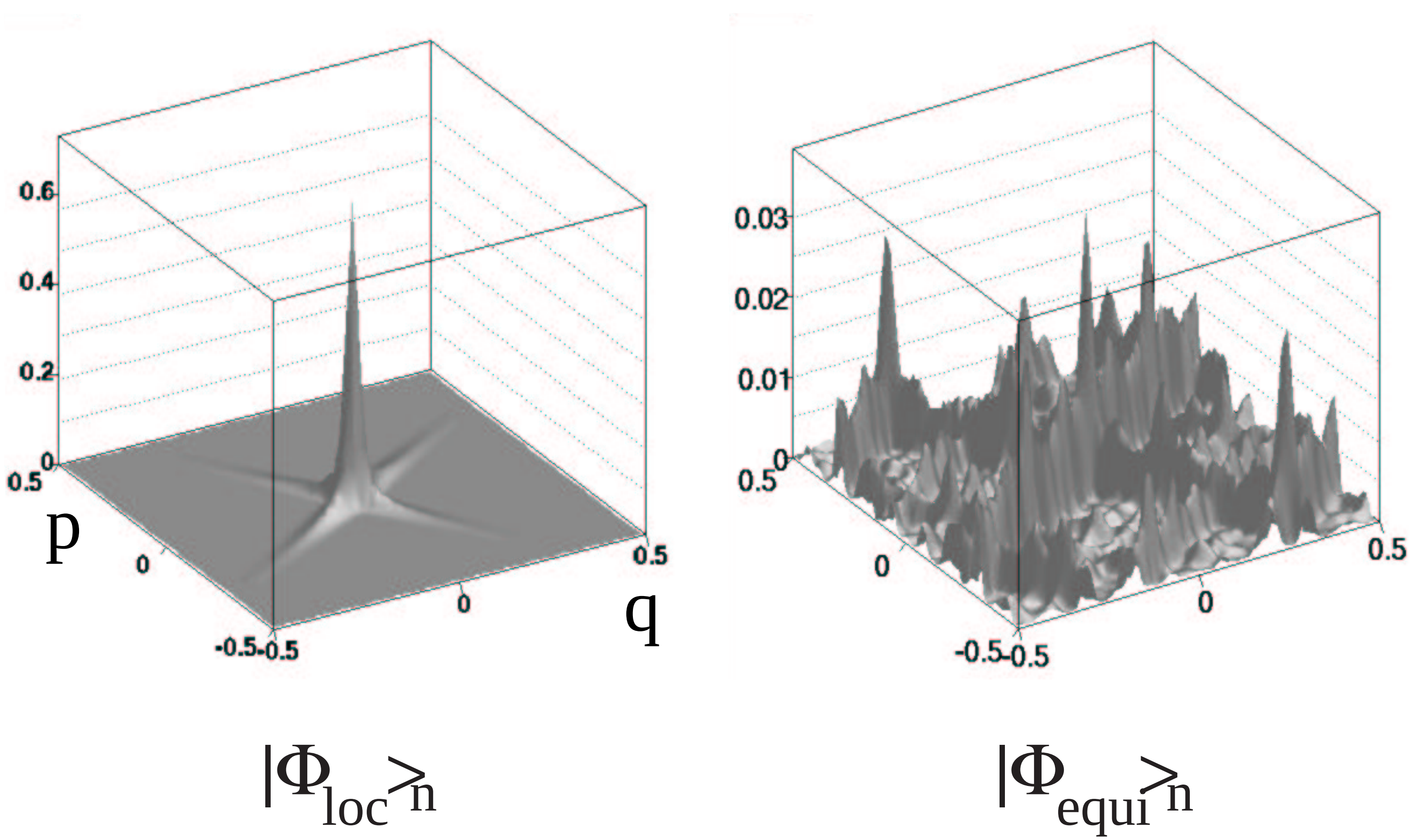
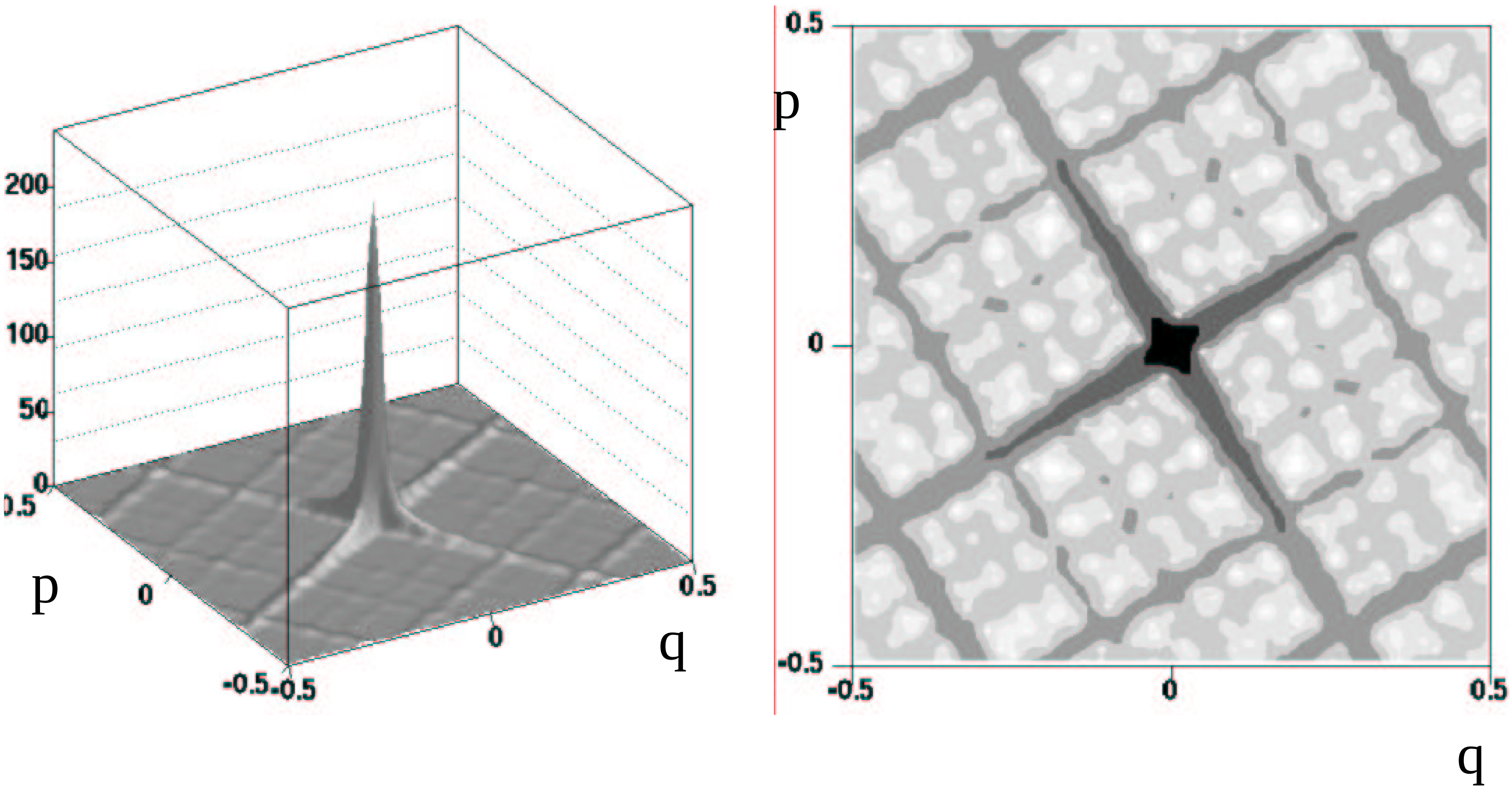


Figure 2: Quasimode $|\Phi_\phi\rangle$ at the origin for N=500, $\phi = 0$: linear (left) and logarithmic (right) plots of the Husimi function



Periodicity of quantum cat maps

Each quantum cat map $\hat{M}_\hbar$ is a periodic matrix [Hannay-Berry,Keating]: $\forall \hbar = \frac{1}{2\pi N}$, there is a *quantum period* $P(\hbar)$ s.t.

$$\hat{M}_\hbar^{P(\hbar)} = e^{i\varphi(\hbar)}\, \hat{1}_{\mathcal{H}_N}.$$

$\Longrightarrow$ eigenvalues $\phi_j = \frac{\varphi(N)+2\pi j}{P(N)}$, with degeneracies $\simeq \frac{N}{P(N)}$.

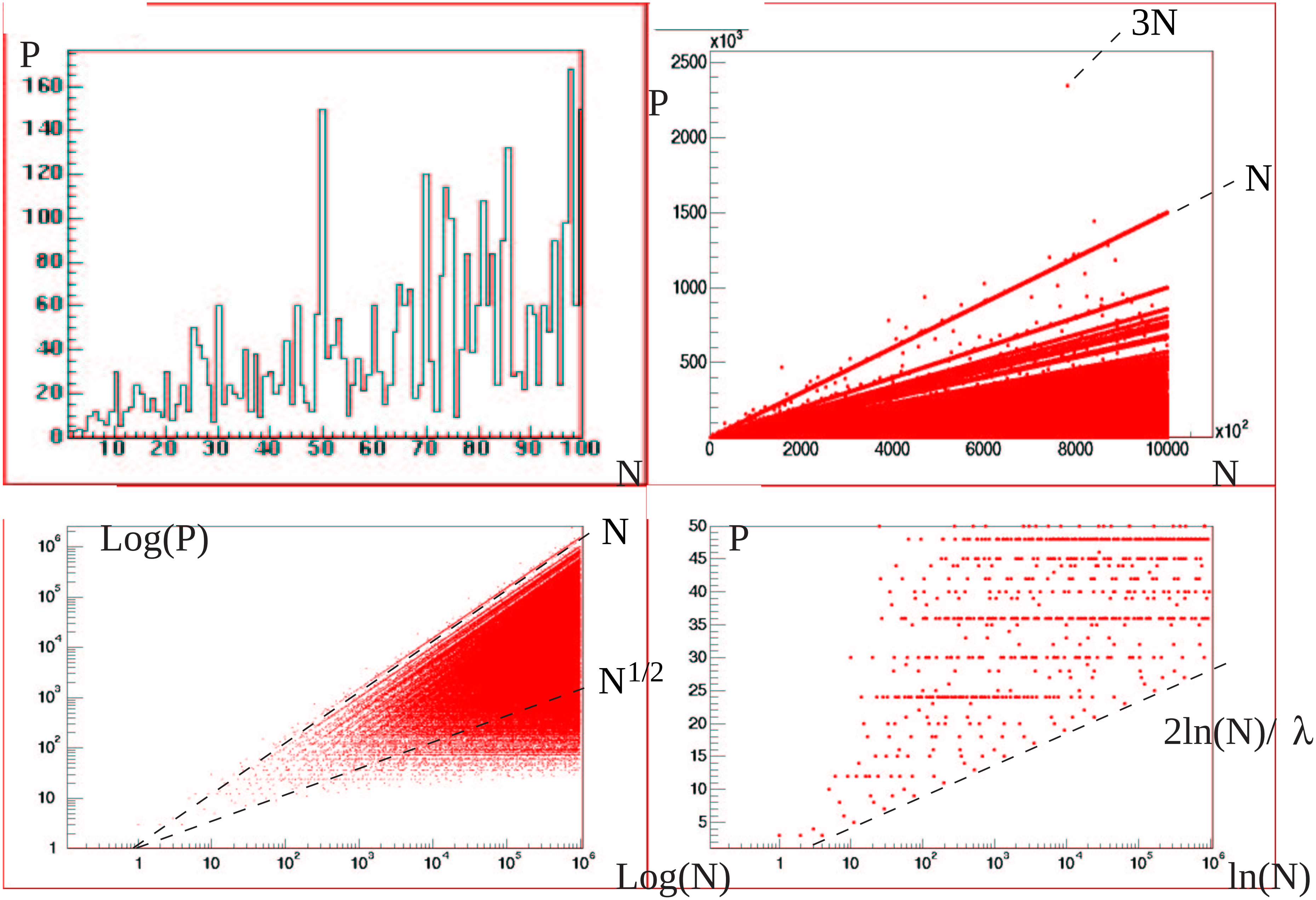
Proposition. *[Kurlberg-Rudnick]*

– *for all integers* N, $c\log N \leq P(N) \leq C\,N \log\log N$.

– $P(N) \geq N^{1/2}$ *for almost all integers* $\Longrightarrow$ *QUE for these integers.*

From our discussion on the correlation function $C(t)$, we must have $P(N) \gtrsim 2T_E(N)$.

• One can construct an infinite explicit (sparse) sequence $\{N_k\}$ s.t. $P(N_k) = 2T_E + \mathcal{O}(1)$: "**short periods**" [Bonechi-DeBièvre]


P
N
3N
N
P
N
Log(P)
N
$N^{1/2}$
Log(N)
P
2ln(N)/ λ
ln(N)

From quasimodes to eigenstates

Proof of Thm. 1

If ϕ_j is an eigenvalue of $\hat{M}_\hbar$, then

$$\hat{\Pi}_{\phi_j} = \frac{1}{P(\hbar)} \sum_{t=t_0}^{t_0+P(\hbar)} \mathrm{e}^{-\mathrm{i}\phi_j t}\, \hat{M}_\hbar^t$$

is the **spectral projector** for this eigenvalue.

In the case of a "short period" $P(\hbar_k) \simeq 2T_E(\hbar_k)$,

this projector is the operator we used to construct the quasimode $|\Phi_\phi\rangle$.

$\Longrightarrow$ for x_0 on a periodic orbit $\mathcal{P}$, the projection of the coh. state $|x_{0,\hbar_k}\rangle_{\mathbb{T}^2}$ onto *any* eigenspace of $\hat{M}_{\hbar_k}$ yields a sequence of eigenstates $\{|\Phi_\phi\rangle\}$ satisfying:

$\rho_{\Phi_\phi} dx \to \frac{dx+\delta_{\mathcal{P}}}{2}$ (remainder $= \mathcal{O}\big(|\log\hbar|^{-1/2}\big)$).

We also control $\|\rho^{(norm)}_{\Phi_\phi}\|_{L^s} \sim \frac{C(s,\phi/\lambda)}{\hbar^{1-\frac{1}{s}}|\log\hbar|}$, for any $1 < s \leq \infty$. □

Perspectives

- Thm. 2shows that $\mathfrak{M}_{sc}$ is a nowhere dense closed subset of $\mathfrak{M}$. Can $\mathfrak{M}_{sc}$ contain measures with a Lebesgue component $< 1/2$? Can a semiclassical measure have a small entropy ?

- The spectral degeneracy $\sim \frac{N}{\log N}$ is a **non-generic feature**, seems to disappear for (nonlinear) perturbations of the cat map of type $\mathrm{e}^{-\mathrm{i}\epsilon \hat{H}_{\mathbb{T}^2}/\hbar} \circ \hat{M}_\hbar$. Strong scarring of eigenstates is unlikely for pert. cat maps.

- Control the time evolution of localized states for perturbed cat maps up to (or beyond) Ehrenfest time (cf. R. Schubert's work on hyperbolic surfaces)

$\rightarrow$ prove a "transition localized $\rightarrow$ equidistributed"
$\rightarrow$ constrain $\mathfrak{M}_{sc}$ for nonlinear maps?