GRENNADY LYUBEZNIK (p.1) 02-05

"Local Cohomology and Its Applications", Marcel Dekker 2001

J. Greenlees "Local Cohom. in Equivariant top."

R = comm. Noeth. ring; $I = (f_1, \dots, f_s)$ ideal of R

$H_I^i(M) = H^i(0 \to M \to \bigoplus_j M_{f_j} \to \bigoplus_{j<k} M_{f_j f_k} \to \cdots \to M_{f_1 \cdots f_s} \to 0)$

$H_I^i(M)$ depend only on $\sqrt{I}$

Application: $k = \bar{k}$; $V \subseteq \mathbb{A}_k^n$ alg. set

i.e., V = sol. ~~set~~ sp. of $(f_1 = 0, f_2 = 0, \dots, f_s = 0)$ where $f_i \in R = k[X_1, \dots, X_n]$.

Q: Given V, what is the minimum s?

A: $s \geq$ height $I = I(V)$ in R

A: If $H_I^i(M) \neq 0 \Rightarrow s \geq i$

Example: $V \subset \mathbb{A}_{\mathbb{C}}^6$ is the solution space of $\Delta_1 = 0, \Delta_2 = 0, \Delta_3 = 0$, where Δ_i are the 2×2 minors of $\begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \\ X_{31} & X_{32} \end{pmatrix}$; $\Delta_i \in R = \mathbb{C}[X_{ij}]$

ht $I = 2 \Rightarrow s \geq 2$

FACT: $H_I^3(R) \neq 0 \Rightarrow$ minimum s is 3

FACT: Given R, I, M, there is no known algorithm to tell if $H_I^i(M) = 0$.

FACT: If $R = k[X_1, \dots, X_n]$ and $M = R$, there is an algorithm!!!

(GL p 2)

Char 0 algorithm due to U. Walther (1999)

$D = R\langle \partial_1, \dots, \partial_n \rangle \subset \mathrm{Hom}_k(R, R)$ $\quad \partial_i = \partial/\partial x_i$

FACT: R is a D-module

FACT: M a D-module $\Rightarrow M_f$ is a D-module

$$\partial_i(m/f) = \frac{f\partial_i(m) - \frac{\partial f}{\partial x_i}\cdot m}{f^2}$$

$\Rightarrow H^i_I(R)$ is a f.g. D-module

II. Let A be a local ring containing a field.

$R \xrightarrow{\varphi} A$ a surjection with R regular local.

$I = \mathrm{Ker}\,\varphi$; $n = \dim R$; k = residue field of A

FACTS ① $\lambda_{q,i} = \dim_k \mathrm{Ext}^i_R(k, H^{n-q}_I(R))$ all finite

② $\lambda_{q,i}$ depend only on A ⌈ Pf. in Char 0 uses D-mod.

Known: ① $\lambda_{q,i} = 0$ if $i > d = \dim A$ or if $q > i$

② $\lambda_{d,d} \neq 0$ $(d = \dim A)$

$\lambda_{q,i}$ seem to be connected to the top. of Spec A

FACT: $V \subseteq \mathbb{P}^n_{\mathbb{C}}$ is smooth; A = hom. coord. ring of V localized at the irrelevant ideal.

Then $\lambda_{0,i}(A) = \dim_{\mathbb{C}} H^i_{sing}(V; \mathbb{C})$

Open Problem: $\lambda_{0,i}(A)$ depend only on V; not on the embedding

(GL p3)

Recent theorem: Assume char $k = p > 0$. Let Γ be the graph on the minimal primes of $\hat{A}$, such that P, Q are connected by an edge iff $ht(P+Q) = 1$. Then $\lambda_{d,d}$ = # of connected components of Γ.

III. Topology of varieties of small codimension in $\mathbb{P}^n_{\mathbb{C}}$.

Theorem 1 Let $V \subset \mathbb{P}^n_{\mathbb{C}}$ be an algebraic set consisting of irreducible components of codim $\leq b$. Then $H_i(\mathbb{P}^n_{\mathbb{C}}, V; \mathbb{Z}) = 0$ and $\pi_i(\mathbb{P}^n_{\mathbb{C}}, V) = 0$, provided ① $i \leq [n/b] - 1$, ② $i \leq [(n-1)/b]$, V is irreducible

(① proved by Peternell, 1980, via analysis)
(② proved by —, 1993, via Local étale cohomology)

Theorem 2 Let R be a complete regular local ring containing a field, with an algebraically closed residue field. Let $n = \dim R$ and $I \subset R$ an ideal whose minimal primes have height $\leq b$. Then

① $H^i_I(R) = 0 \ \forall i > n - [(n-1)/b]$ * (Faltings, 1980)
② $H^i_I(R) = 0 \ \forall i > n - 1 - [(n-2)/b]$ ** if I is prime (Huneke-L, 199[illegible])

FACT: * = ** iff $b \mid (n-1)$
Q: Assume $b \nmid (n-1)$ (so * = 1 + **). When is $H_I^{n-[(n-1)/b]}(R) = 0$?

Recent Theorem: Let Δ be the simplicial complex on the min. primes $P_1, \ldots, P_s$ of I, s.t. $(P_{i_1}, \ldots, P_{i_j}) \subset \Delta$ iff $\sqrt{(P_{i_1} + \ldots + P_{i_j})} \neq \mathfrak{m}$. Then $H_I^{n-[(n-1)/b]} = 0$ iff $\tilde{H}^{[(n-1)/b]-1}(\Delta; k) = 0$

$\lambda_{g,i} = 0$ if $g > 0$, but $\lambda_{g,d}$ may not be 0