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Commutative algebra of N points in the plane

Notes for these lectures are at
http://www.math.berkeley.edu/~mhaiman

Suppose $\underset{(x_1,y_1)}{P_1}, \dots, \underset{(x_n,y_n)}{P_n} \in \mathbb{C}^2$. Let $E = \mathbb{C}^{2n}$

$C[E] = \mathbb{C}[x_1, y_1, \dots, x_n, y_n]$.

Coincidence locus $V_{ij} = V(x_i - x_j, y_i - y_j)$

$V = \bigcup_{i<j} V_{ij}$ $\quad I(V) = \bigcap_{i<j} (x_i - x_j, y_i - y_j)$

Problem: Describe I.

Case $\underset{x_1}{P_1}, \dots, \underset{x_n}{P_n} \in \mathbb{C}^1$ points on a line

$J = \bigcap_{i<j} (x_i - x_j)$

Observe:

1) $J = (\Delta(\underline{x}))$

where $\Delta(x) = \prod_{i<j} (x_i - x_j) = \det \underbrace{\begin{bmatrix} 1 & x_1 & \cdots & x_1^{n-1} \\ \vdots & & & \\ 1 & x_n & \cdots & x_n^{n-1} \end{bmatrix}}_{\text{Vandermonde}}$

2) J is a free $\mathbb{C}[\underline{x}]$-module

3) $J^m = (\Delta(\underline{x})^m) = \bigcap_{i<j} (x_i - x_j)^m = J^{(m)}$

4) Rees algebra $\mathbb{C}[\underline{x}][tJ]$ is a polynomial ring.

all follows because $V(J)$ is a hyperplane arrangement
higher dim: more general subspace arrangement

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Now what happens on plane?

Consider (1)

$D \subseteq \mathbb{N} \times \mathbb{N}$ $|D| = n$ $D = \{(\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)\}$

$$\Delta_D(\underline{x}, \underline{y}) = \det \begin{bmatrix} x_1^{\alpha_1} y_1^{\beta_1} & \cdots & x_1^{\alpha_n} y_1^{\beta_n} \\ \vdots & & \vdots \\ x_n^{\alpha_1} y_n^{\beta_1} & \cdots & x_n^{\alpha_n} y_n^{\beta_n} \end{bmatrix}$$

$\sigma \in S_n$; $\sigma x_i = x_{\sigma(i)}$, $\sigma y_i = y_{\sigma(i)}$

$\sigma \Delta_D(\underline{x}, \underline{y}) = \mathrm{sgn}(\sigma) \Delta_D$

$\Rightarrow \Delta_D \in \mathbb{C}[\underline{x}, \underline{y}]^{\varepsilon}$ space of alternating polynomials

in fact, $\{\Delta_D : \text{all } D\}$ is a vector space basis for $\mathbb{C}[x, y]^{\varepsilon}$

<u>Theorem 1</u> $I = (\Delta_D : \text{all } D)$

minimal generators?

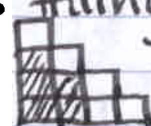
I is doubly-graded by degree in x & degree in y

$M = I/(x, y)I$

<u>Theorem 2</u> $\dim M = C_n = \frac{1}{n+1}\binom{2n}{n}$, the nth Catalan #.

$M = \bigoplus_{r,s} M_{r,s}$ $C_n = \# \lambda \subseteq \delta$, $\delta = (n-1, n-2, \ldots, 1)$

= # of partitions fitting inside a staircase:



δ

$\lambda = (2,2,1)$

$a(\lambda) = |\delta| - |\lambda|$

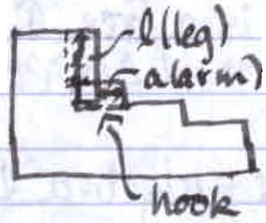
$b(\lambda) = $ # hooks in λ s.t. $a \in \{l, l+1\}$

a = length of arm

hook: pick a square ~~on border~~

vertically above it is its leg

horizontally right of it is its arm



<u>Thm 3</u> (Garsia-Haglund-Haiman)

$$\sum_{r,s} q^r t^s \dim M_{r,s} = \sum_{\lambda \subseteq \delta} q^{a(\lambda)} t^{b(\lambda)}$$

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② a of 2b

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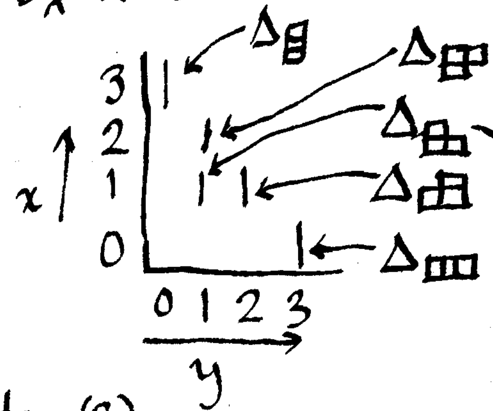
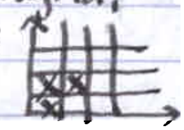
Problem Find D_λ for each $\lambda \leq \delta$
s.t. $\deg_x \Delta_{D_\lambda} = a(\lambda)$, $\deg_y \Delta_{D_\lambda} = b(\lambda)$

$I = (D_\lambda : \lambda \leq \delta)$

$n=3$



← means on $n \times n$ grid, the 3pts are: these 3



$$\det \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}$$

Consider (2)

Theorem 4 I is a free $\mathbb{C}[\underline{x}]$-module
(Hence a free $\mathbb{C}[\underline{x}, y_1]$-module.)
depth $I = n+1$.

Consider (3)

Theorem 5 $I^m = \bigcap_{i<j} (x_i - x_j, y_i - y_j)^m$ for all m.

Theorems 1, 4, and 5 follow from
Theorem 6 $\forall m$ $(\Delta_D : \text{all } D)^m$ is a free $\mathbb{C}[\underline{x}]$-module.

Corollary $(\Delta_D : \text{all } D)^m = I^{(m)}$

Pf of Cor from Thm

Step 0: $\subseteq$ is clear

Step 1: localize: locally at $\underline{P}$ where $P_i \neq P_j$, both sides factor. So $(\Delta_D)^m_{\underline{P}} = I^{(m)}_{\underline{P}}$ by induction on n.

Step 2: Thm 6 $\Rightarrow$ $\mathbb{C}[\underline{x}, \underline{y}]/(\Delta_D)^{(m)}$ has depth $\geq n-1$ as a $\mathbb{C}[\underline{x}]$-module.

$\Rightarrow$ no associated prime supported in $V(x_1 - x_2, \ldots, x_{n-1} - x_n)$

$f \in (\Delta_D)^m_{Q_m}$ for $Q \in \operatorname{Spec} \mathbb{C}[\underline{x}] \Rightarrow f \in (\Delta_D)^m$

(Holds for $f \in I^{(m)}$ by Step 1.) $\square$

Consider (4)

Rees algebra $R = \mathbb{C}[\underline{x}, \underline{y}][t(\Delta_D)]$

$X = \operatorname{Proj} R$

$$\begin{array}{ccc} X & \to & E \\ \downarrow & & \downarrow \\ X/S_n & \to & E/S_n \end{array}$$

<u>Prop</u> $X/S_n = \operatorname{Hilb}^n(\mathbb{C}^2)$

the Hilbert scheme of points in the plane

$= \{ I \subseteq \mathbb{C}[x,y] : \dim_{\mathbb{C}} \mathbb{C}[x,y]/I = n \}$

<u>Thm</u> (Fogarty, 60's) $\operatorname{Hilb}^n(\mathbb{C}^2)$ is irreducible & nonsingular.

true only of $\mathbb{C}^2$, makes things work out nicely

Remark codim "y-axis" in $\operatorname{Hilb}^n(\mathbb{C}^2)$ is n

In particular, $\dim R/(y) = n+1 = \dim R/(x)$

<u>Thm 7</u> R is Gorenstein (X is arithmetically Gorenstein)

this implies Theorem 6

however, the existing proof assumes Theorem 6

<u>Problems:</u>

- Prove Thm 6 and/or Thm 7 directly
- Improve Thm 7. Does R have rational singularities?
- ~~our~~ $V' = \bigcup_{i<j} V(x_i - x_j)$ is a hyperplane arrangement in $\mathbb{C}^n$

  $V = \mathbb{C}^2 \otimes V' = \bigcup_{i<j} \mathbb{C}^2 \otimes V(x_i - x_j) \subseteq \mathbb{C}^2 \otimes \mathbb{C}^n$

  What about $I = I(\mathbb{C}^d \otimes V')$?

  Also for other hyperplane arrangements?

  (must be a <u>free</u> hyperplane arrangement)