Ruchira Datta MSRI Comm. Alg. Workshop Notes

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The geometry of syzygies

roots in invariant theory - Hilbert

let $S = K[x_0, \dots, x_r]$, $m = (x_0, \dots, x_r)$

$\overset{\swarrow \text{invariant ring}}{S^G} = K[f_1, \dots, f_t] \twoheadleftarrow K[y_1, \dots, y_t]$

Hilbert wrote formulas counting these

$\overset{\swarrow\text{variety}}{X} \subset \mathbb{P}^r \leftrightsquigarrow S_X = S/I_X$, its coord ring

family of varieties

$\mathcal{X} \subset \mathbb{P}^r \times \mathbb{A}^s \leftrightsquigarrow S_{\mathcal{X}} = S[t_1, \dots, t_s]/I$

$X_p \leftrightsquigarrow S_{\mathcal{X}}|_{t=t(p)}$

$S_X \rightsquigarrow$ Hilbert ~~series~~ function $H_{S_X}(d) = \dim_K (S_X)_d$

<u>Thm</u> $H_{S_X}(d) \underset{d \gg 0}{=} P_X(d)$, a polynomial ($= \chi \mathcal{O}_X(d)$)

$\mathcal{X}$ is <u>flat</u> $\Leftrightarrow P_{X_p}(d)$ is identically $\equiv P(d)$ (independent of p)

<u>Thm</u> The family S_{X_p} is flat $\Leftrightarrow H_{S_X}(d)$ is constant

<u>Thm</u> Let $M = \bigoplus M_d$ be a finitely generated (f.g.) graded S-module

Then $H_M(d) \triangleq \dim_K M_d$ is polynomial for large d, and more...

<u>Proof</u> (Hilbert) Let $S(-a)$ be the rank 1 free module with generator in degree a. (This makes the formula $M(b)_d = M_{b+d}$ work out.)

Looking at the combinatorics shows

$H_S(d) = \binom{r+d}{r}$ – polynomial for $d \geq -r$

Then $H_{S(-a)} = \binom{r+d-a}{r}$

$$0 \to \underset{\substack{\| \\ F_t}}{\bigoplus_j S(-a_{tj})} \to \cdots \to \underset{\substack{\| \\ F_1}}{\bigoplus_j S(-a_{1j})} \to \underset{\substack{\| \\ F_0}}{\bigoplus_j S(-a_{0j})} \to M \to 0$$

F_0 – free modules

Hilbert Syzygy Theorem says: $t \leq r+1$

<u>Corollary</u> $H_M^{(d)} = \sum_{i=0}^{t} (-1)^i \sum_j \binom{r+d-a_{ij}}{r}$

Write $F_i = \bigoplus_j S(-j)^{\beta_{ij}}$ (a different j)

i.e., β_{ij} copies of the degree j piece

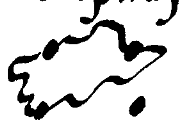
$\{\beta_{ij}\}$ are the "graded Betti numbers"

$H_M^{(d)} = \sum_i (-1)^i \sum_j \beta_{ij} \binom{r+d-j}{r}$

<u>Example</u> X = 4 distinct points in $\mathbb{P}^2$

$\mathcal{X} \subset \mathbb{P}^2 \times \underbrace{(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 - \text{diagonals})}_{T} \to T$

for basis of dim 4, (in each graded piece)
take high degree curve going through
3pts & not 4th:



⇒ nonzero on one pt,
zero on others

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2 a of 3b

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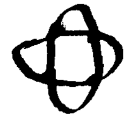
So $P_{X_p}(d) = 4$

no 3 on a line – then up to PGL(4),
the 4 pts are $(1,0,0)$
$(0,1,0)$ $(1,1,1)$
$(0,0,1)$

$X = 4$ pts $\subset \mathbb{P}^2$
Cases: 1) No 3 on a line



2 conics

or conics as pairs of lines

$H_{S_X}(d) \backslash d$	0	1	2	3
	1	3	4	4

how many cubic forms
are there? $\binom{2+3}{2} = 10$

could get cubic forms as
products of linear forms & quadratic forms
$l_1 Q_1 = l_2 Q_2$? ~~do these overlap?~~
– then would have common divisor

$$0 \to S(-4) \xrightarrow{\begin{pmatrix} Q_2 \\ -Q_1 \end{pmatrix}} S(-2)^2 \xrightarrow{(Q_1, Q_2)} S \to S_X \to 0$$

$\beta_{24} = 1$ $\beta_{12} = 2$ $\beta_{00} = 1$

the rest of the Betti numbers are zero

<u>Betti diagram</u> $S \leftarrow S(-2)^2 \leftarrow S(-4)$

0	1	–	–
1	–	2	–
2	–	–	1

β_{00} β_{12} β_{24}

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Digression: Minimality

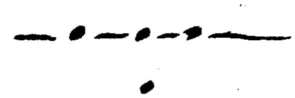
A complex of free modules $\cdots \to F_{i+1} \xrightarrow{\varphi_{i+1}} F_i \xrightarrow{\varphi_i} F_{i-1} \to \cdots$
is minimal if $\varphi_i(F_i) \subset m F_{i-1}$

in general the Betti diagram looks like:

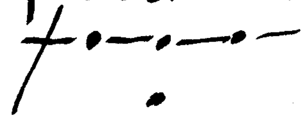
$$\begin{array}{llll} M & & & \\ i & \beta_{0,i} & \beta_{1,i+1} & \cdots \\ i+1 & \vdots & \vdots & \ddots \\ i+2 & \vdots & \vdots & \ddots \end{array}$$

Cases

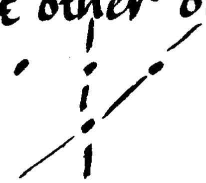
3 points on a line, one off



quadratic function vanishing on
3pts but not off pt: product of linear forms

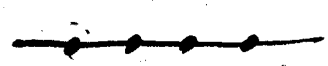


2 on pts & off pt, but not other on pt:



$$H_{S_X}(d) \quad \begin{array}{|cccc} 0 & 1 & 2 & 3 \\ \hline 1 & 3 & 4 & 4 \end{array}$$

4 points on a line



$$\begin{array}{lcccc} & 0 & 1 & 2 & 3 \\ H_{S_X} & 1 & 2 & 3 & 4 \end{array}$$

a quadric containing 3 collinear
pts also contains the 4th

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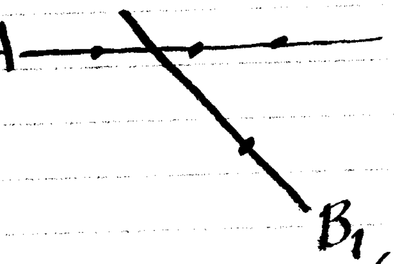
back to:
<u>3 pts on a line, one off</u>

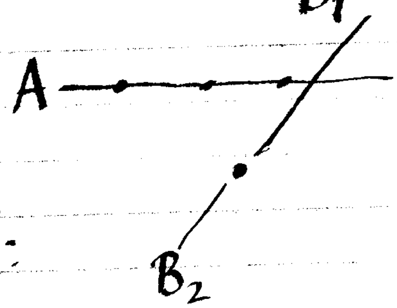
quadrics vanishing on all 4 pts

product of linear forms

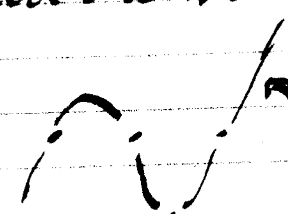


AB_1, AB_2

have syzygy:
$B_2(AB_1) = B_1(AB_2)$



⇒ these don't give all cubics, need one more to generate:



$C = F_1B_1 + F_2B_2$

$$0 \longrightarrow \begin{matrix} S(-3) \\ \oplus \\ S(-4) \end{matrix} \xrightarrow{\begin{pmatrix} B_2 & F_1 \\ -B_1 & F_2 \\ 0 & -A \end{pmatrix}} \begin{matrix} S^2(-2) \\ \oplus \\ S(-3) \end{matrix} \xrightarrow{(AB_1, AB_2, C)} S$$

Betti diagram:

1		
	2	1
	1	1

Hilbert function may be written as generating function

$$\sum H_M(d)t^d = \frac{\cdots}{(1-t)^r}$$

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Betti diagram for 4 pts on a line

1	1	-
-	-	-
-	-	-
-	1	1