Kähler Fibrations
and
Complex Finsler Geometry

Tadashi Aikou

In this talk, we shall investigate complex Finsler geometry from the view point of Kähler fibrations. If a convex Finsler metric F is given on a holomorphic vector bundle $\pi_E : E \to M$, then its projective bundle $\pi_{\mathbb{P}(E)} : \mathbb{P}(E) \to M$ is a Kähler fibration with the pseudo-Kähler metric $\varPi_{\mathbb{P}(E)} = \sqrt{-1}\partial\bar{\partial}\log F$. We shall investigate the flatness of the metrical Bott connection $D^{\mathbb{P}(E)}$ naturally defined by $\varPi_{\mathbb{P}(E)}$, and the projective-flatness of F.

Contents

Kähler Fibrations

M: connected complex manifold of $\dim_{\mathbb{C}} M = n$

$\mathcal{X}$: connected complex manifold of $\dim_{\mathbb{C}} \mathcal{X} = n + r$

$\pi_{\mathcal{X}} : \mathcal{X} \to M$: holomorphic submersion

$(z^1, \cdots, z^n, \xi^1, \cdots, \xi^r)$: local coordinate system on $\mathcal{X}$

$(z^1, \cdots, z^n)$: local coordinate system on M

- **Kähler fibration** $\pi_{\mathcal{X}} : \mathcal{X} \to M$

$\overset{\text{def.}}{\Longleftrightarrow}$

$X_z = \pi^{-1}(z)$: connected Kähler manifold with Kähler metric

$$\Pi_z = \sqrt{-1}\partial\bar{\partial}G_z$$

The metric on X_z is given by

$$\left\langle \frac{\partial}{\partial \xi^i}, \frac{\partial}{\partial \xi^j} \right\rangle = G_{i\bar{j}}(z, \xi) = \frac{\partial^2 G_z}{\partial \xi^i \partial \bar{\xi}^j}$$

(parameterized by $z \in M$ smoothly)

- **Fundamental sequence**

$$\mathbb{O} \to \mathcal{V}_{\mathcal{X}} \xrightarrow{i} T_{\mathcal{X}} \xrightarrow{d\pi_{\mathcal{X}}} \pi_{\mathcal{X}}^{-1} T_M \to \mathbb{O}$$

where

$$\mathcal{V}_{\mathcal{X}} = \ker d\pi_{\mathcal{X}} \ (\textbf{vertical subbundle})$$

- **Connection**

$h_{\mathcal{X}} : \pi^{-1} T_M \to T_{\mathcal{X}}$ such that

$$T_{\mathcal{X}} = \mathcal{V}_{\mathcal{X}} \oplus \mathcal{H}_{\mathcal{X}}$$

where

$$\mathcal{H}_{\mathcal{X}} = h_{\mathcal{X}}\left(\pi_{\mathcal{X}}^{-1} T_M\right) \ (\textbf{horizontal subbundle})$$

$X_\alpha = h_{\mathcal{X}}(\partial/\partial z^\alpha)$: **horizontal lift**

$\overset{\text{def}}{\Longleftrightarrow}$

$$X_\alpha = \frac{\partial}{\partial z^\alpha} - \sum N_\alpha^i \frac{\partial}{\partial \xi^i}$$

$N_\alpha^i(z, \xi)$: **canonical connection**

$\overset{\text{def}}{\Longleftrightarrow}$

$$\boxed{N_\alpha^i = \sum G^{i\bar{m}} \frac{\partial^2 G}{\partial z^\alpha \partial \bar{\xi}^m}}$$

- **Bott connection** $D^{\mathcal{X}}$ associated with the canonical connection $h_{\mathcal{X}}$

  $D^{\mathcal{X}} : \mathcal{V}_{\mathcal{X}} \to \mathcal{V}_{\mathcal{X}} \otimes \Omega^1_{\mathcal{X}}$

  $\overset{\text{def}}{\Longleftrightarrow}$

$$D^{\mathcal{X}}_X Y = [X, Y]^V, \quad X \in \mathcal{H}_{\mathcal{X}}, \ Y \in \mathcal{V}_{\mathcal{X}}$$

  $\Gamma^i_{j\alpha}(z, \xi)$: connection coefficients

  $\Longrightarrow$

$$\boxed{\Gamma^i_{j\alpha} = \frac{\partial N^i_\alpha}{\partial \xi^j}}$$

- **Facts**

Proposition 1

$D^{\mathcal{X}}$ satisfies

$$d^h_{\mathcal{X}} \langle Y, Z \rangle = \left\langle D^{\mathcal{X}} Y, Z \right\rangle + \left\langle Y, D^{\mathcal{X}} Z \right\rangle$$

Theorem 1

The following conditions are equivalent

(1) $h_{\mathcal{X}}$ is flat (i.e., $\mathcal{H}_{\mathcal{X}}$ is holomorphic and integrable).

(2) $\partial^h_{\mathcal{X}} = \sum \frac{\partial}{\partial z^\alpha} \otimes dz^\alpha$ (i.e., $N^i_\alpha = 0$) w.r.t. ${}^\exists(z^\alpha, \xi^i)$

(3) $D^{\mathcal{X}}$ is flat (i.e., $D^{\mathcal{X}} \circ D^{\mathcal{X}} \equiv 0$)

Finsler Vector Bundles

$\pi_E : E \to M$:holomorphic vector bundle of $\mathrm{rank}(E) = r$

$\{U, s_U = (s_1, \cdots, s_r)\}$: local holomorphic frame fields of E

(z^α, ξ^i): local coordinate on $\pi_E^{-1}(U)$ defined by $\{U, s_U\}$

- **Convex Finsler metric** $F : E \to \mathbb{R}$

$\overset{\text{def}}{\Longleftrightarrow}$

(1) $F(z,\xi) \geq 0$, and $F(z,\xi) = 0$ iff $\xi = 0$

(2) F is smooth on $E^\times = E \backslash \{0\}$

(3) $F(z, \lambda\xi) = |\lambda|^2 F(z,\xi)$

(4) $F(z,\xi)$ is strongly pseudo-convex on each E_z, i.e.,

$$F_{i\bar{j}}(z,\xi) = \frac{\partial^2 F_z}{\partial \xi^j \partial \bar{\xi}^j} > 0$$

$\Longrightarrow$ E_z: Kähler manifold with

$$\left\langle \frac{\partial}{\partial \xi^i}, \frac{\partial}{\partial \xi^j} \right\rangle = F_{i\bar{j}}(z,\xi)$$

$\Longrightarrow$ $\pi_E : E \to M$: Kähler fibration with

$$\Pi_E = \sqrt{-1}\partial\bar{\partial}F$$

- **Non-linear connection**

$h_E : \pi^{-1}T_M \to T_E$

$\overset{\text{def}}{\Longleftrightarrow}$

$$h_E\left(\frac{\partial}{\partial z^\alpha}\right) = \frac{\partial}{\partial z^\alpha} - \sum N^i_\alpha \frac{\partial}{\partial \xi^i} := X_\alpha$$

with

$$\boxed{N^i_\alpha = \sum F^{i\bar{m}} \frac{\partial^2 F}{\partial z^\alpha \partial \bar{\xi}^m}}$$

- **Special Finsler metrics**

Definition

(1) (E, F): **flat** (or **locally Minkowski**)

$\overset{\text{def}}{\Longleftrightarrow}$

$\boxed{F_{i\bar{j}} = F_{i\bar{j}}(\xi)}$ w. r. t. ${}^\exists(U, s_U)$.

(2) (E, F): **Berwald** (or **modeled on** $\cdots$)

$\overset{\text{def}}{\Longleftrightarrow}$

$\boxed{h_E : \text{ linear}}$

- **Bott connection** D^E

$$D^E_{X_\alpha}\frac{\partial}{\partial\xi^j}=\sum\Gamma^i_{j\alpha}\frac{\partial}{\partial\xi^i}$$

where

$$\boxed{\Gamma^i_{j\alpha}=\frac{\partial N^i_\alpha}{\partial\xi^j}}$$

- **Torsion** T of D^E

$\overset{\text{def}}{\Longleftrightarrow}$

$$T^i=d\theta^i+\sum\omega^i_j\wedge\theta^j\quad(\theta^i:\text{dual frame for }\mathcal{V}_E)$$

Definition

(1) $R^i_{\alpha\bar\beta}=-\overline{X_\beta}N^i_\alpha \implies$ (**integrability tensor** for $\mathcal{H}_E$)

(2) $R^i_{\alpha\bar j}=-\partial N^i_\alpha/\partial\bar\xi^j$

$\Longrightarrow$

$$T^i=\sum\bar\partial N^i_\alpha\wedge dz^\alpha=\sum R^i_{\alpha\bar\beta}dz^\alpha\wedge d\bar z^\beta+\sum R^i_{\alpha\bar j}dz^\alpha\wedge\bar\theta^j$$

- **Curvature** Ω^E of D^E

$\Longrightarrow$

$$\Omega^E=d^h_E\omega+\omega\wedge\omega=\bar\partial^h_E\omega$$

$$R^i_{j\alpha\bar\beta}=\frac{\partial R^i_{\alpha\bar\beta}}{\partial\xi^j}-\sum R^i_{\alpha\bar m}\overline{R^m_{\beta\bar j}}$$

- **Facts**

Proposition 2

(1) (E, F) is Berwald if and only if $R^{i}_{\alpha\bar{j}} = 0$

(2) If (E, F) is Berwald, then there exists a connection $\Gamma^{i}_{j\alpha}(z)$ such that

$$N^{i}_{\alpha} = \sum \Gamma^{i}_{j\alpha}(z)\xi^{j}$$

(this connection $\Gamma^{i}_{j\alpha}(z)$ is the Hermitian connection of a Hermitian metric g_F on E cf. [Ai1])

Theorem 2(cf. [Ai6])

(E, F) is flat if and only if

(1) D^E is flat, i.e., $R^{i}_{j\alpha\bar{\beta}} = 0$

or

(2) (E, F) is Berwald and its associated Hermitian metric g_F is flat.

Projective Finsler Bundles

Projective bundle $\pi_{\mathbb{P}(E)} : \mathbb{P}(E) = E^{\times}/\mathbb{C}^{\times} \to M$

Natural projection $\rho : E^{\times} \to \mathbb{P}(E)$

(E, F): convex Finsler vector bundle

$\Longrightarrow$ $\mathbb{P}(E_z) = \pi^{-1}_{\mathbb{P}(E)}(z)$: Kähler manifold with

$$\left\langle d\rho\left(\frac{\partial}{\partial \xi^i}\right), d\rho\left(\frac{\partial}{\partial \xi^j}\right)\right\rangle = \frac{\partial^2 \log F}{\partial \xi^i \partial \bar{\xi}^j} := G_{i\bar{j}}(z, \xi)$$

$\Longrightarrow$ $\pi_{\mathbb{P}(E)} : \mathbb{P}(E) \to M$: Kähler fibration with

$$\Pi_{\mathbb{P}(E)} = \sqrt{-1}\partial\bar{\partial} \log F$$

(pseudo-Kähler metric on $\mathbb{P}(E)$)

- **Connection**

$h_{\mathbb{P}(E)} : \pi^{-1}_{\mathbb{P}(E)} T_M \to T_{\mathbb{P}(E)}$

$\overset{\text{def}}{\Longleftrightarrow}$

$$\partial^h_{\mathbb{P}(E)} = \sum d\rho(X_\alpha) \otimes dz^\alpha$$

- **Projectively flat Finsler metrics**

Definition

(E, F): **projectively flat**
$\overset{\text{def}}{\Longleftrightarrow}$

$$\boxed{\frac{\partial^2 \log F}{\partial \xi^i \partial \bar{\xi}^j} := G_{i\bar{j}} = G_{i\bar{j}}(\xi)} \qquad \text{w. r. t.} \qquad {}^{\exists}(U, s_U)$$

$\Longleftrightarrow$

$$\boxed{F(z, \xi) = e^{\sigma(z)} G(\xi)} \qquad \text{for} \qquad {}^{\exists}\sigma(z) \in C^{\infty}(U)$$

- **Fact**

Theorem 3
(E, F) is projectively flat if and only if

(1) The Bott connection $D^{\mathbb{P}(E)}$ associated wtih $h_{\mathbb{P}(E)}$ is flat, i.e.,

$$R^i_{j\alpha\bar{\beta}} = \frac{\partial^2 \sigma(z)}{\partial z^\alpha \partial \bar{z}^\beta} \delta^i_j$$

or

(2) (E, F) is Berwald and its associated Hermitian metric g_F is projectively flat.

• **Some comments**

Suppose that $\mathit{\Pi}_{\mathbb{P}(E)} = \sqrt{-1}\partial\bar{\partial}\log F$ is a Kähler metric on $\mathbb{P}(E)$, i.e.,

$$\partial\bar{\partial}\log F = \begin{pmatrix} -\Psi_{\alpha\bar{\beta}} & 0 \\ 0 & G_{i\bar{j}} \end{pmatrix} : \text{positive-definite}$$

where

$$\Psi_{\alpha\bar{\beta}} = \frac{1}{F}\sum F_{m\bar{k}} R^{m}_{j\alpha\bar{\beta}} \xi^{j} \bar{\xi}^{k}$$

(I) If (E, F) is projectively flat

$\Longrightarrow$

$$\Psi_{\alpha\bar{\beta}} = \frac{1}{F(z,\xi)}\sum F_{m\bar{k}} \frac{\partial^2 \sigma(z)}{\partial z^{\alpha}\partial \bar{z}^{\beta}} \delta^{m}_{j} \xi^{j} \bar{\xi}^{k}$$

$$= \frac{1}{F(z,\xi)}\sum \left(F_{m\bar{k}}\xi^{m}\bar{\xi}^{k}\right) \frac{\partial^2 \sigma(z)}{\partial z^{\alpha}\partial \bar{z}^{\beta}}$$

$$= \frac{\partial^2 \sigma(z)}{\partial z^{\alpha}\partial \bar{z}^{\beta}}$$

$\Longrightarrow$ $\mathit{\Pi}_M = \sqrt{-1}\sum\left(-\Psi_{\alpha\bar{\beta}}(z)\right) dz^{\alpha}\wedge d\bar{z}^{\beta}$: Kähler metric on M

$\Longrightarrow$ $\pi_{\mathbb{P}(E)} : \left(\mathbb{P}(E), \mathit{\Pi}_{\mathbb{P}(E)}\right) \to (M, \mathit{\Pi}_M)$: Kähler submersion

(II) Conversely, if $\pi_{\mathbb{P}(E)} : \big(\mathbb{P}(E), \Pi_{\mathbb{P}(E)}\big) \to (M, \Pi_M)$ is a Kähler submersion for some Kähler metric $\Pi_M = \sqrt{-1}\,\Sigma\, g_{\alpha\bar{\beta}}(z) dz^\alpha \wedge d\bar{z}^\beta$ on M

$$\Longrightarrow \begin{cases} \Psi_{\alpha\bar{\beta}} = -g_{\alpha\bar{\beta}}(z) \\ \\ R^i_{\alpha\bar{\beta}} = A_{\alpha\bar{\beta}}(z,\xi)\xi^i \quad (\mathcal{H}_{\mathbb{P}(E)} \text{ is integrable by Watson's theorem}) \end{cases}$$

$$\Longrightarrow \ A_{\alpha\bar{\beta}} = -g_{\alpha\bar{\beta}}(z)$$

$$\Longrightarrow \ R^i_{j\alpha\bar{\beta}} = -g_{\alpha\bar{\beta}}(z)\delta^i_j - \Sigma\, R^i_{\alpha\bar{m}} \overline{R^m_{\beta\bar{j}}}$$

If (E, F) is Berwald, i.e., $R^i_{\alpha\bar{j}} = 0$

$\Longrightarrow \ R^i_{j\alpha\bar{\beta}} = -g_{\alpha\bar{\beta}}(z)\delta^i_j$: (E, F) is projectively flat

Theorem 4

Suppose that (E, F) is Berwald (in particular F is Hermitian) such that $\Pi_{\mathbb{P}(E)} = \sqrt{-1}\partial\bar{\partial}\log F$ is a Kähler metric on $\mathbb{P}(E)$. Then, (E, F) is projectively flat if and only if $\pi_{\mathbb{P}(E)} : \mathbb{P}(E) \to M$ is a Kähler submersion for some Kähler metric on M.

References

[Ai1] Aikou, T., Complex manifolds modeled on a complex Minkowski space, J. Math. Kyoto Univ. **35**(1995), p. 83–111.

[Ai2] ———, A partial connection on complex Finsler bundles and its applications, Illinois J. Math. **42**(1998), p. 481–492.

[Ai3] ———, Conformal flatness of complex Finsler structures, Publ. Math. Debrecen **54**(1999), p. 165–179.

[Ai4] ———, Some remarks on the conformal equivalence of complex Finsler structures, in *Finslerian Geometry:A Meeting of Minds* (edited by P. Antonelli), Kluwer Acad. Publ., 35–52, 2000.

[Ai5] ———, Applications of Bott connection to Finsler geometry, in *Steps in Differential Geometry, Proceedings of the Colloquium on Differential Geometry*(edited by L. Kozma, P. T. Nagy and L. Tam'assy), University of Debrecen(2001), 1–13.

[Ai6] ———, Differential geometry of Kähler fibrations and its applications to Finsler geometry, Far East J. Math. Sci., vol. **4**(2002), 91–117.

[Ai7] ———, Projective flatness of complex Finsler metrics, Preprint.

[Ko1] Kobayashi, S., Negative vector bundles and complex Finsler metrics, Nagoya Math. J. **57**(1975), pp. 153–166.

[Ko2] ———, Differential Geometry of Complex Vector Bundles, Iwanami-Princeton Univ. Press, 1987.

[Ko-Wu] Kobayashi, S. and Wu, H., On holomorphic sections of certain Hermitian vector bundles, Math. Ann. **189**(1970), pp. 1–4.

[Sz] Szabó, Z. I., Positive definite Berwald spaces, Tensor N. S. **35**(1981), p. 25-39.

[Ws] Watson, B., Almost Hermitian submersions, J. Differential Geometry **11**(1976), 147–165.

Appendix

M: smooth manifold

$M_x \cong \mathbb{R}^n_x$: tangent space at $x \in M$

$L : TM \to \mathbb{R}$: **Finsler metric**

$\Longrightarrow$

- $(M_x, \|\cdot\|_x)$: normed linear space with

$$\|y\|_x = L(x, y)$$

- (M_x, g_x): Riemannian manifold with

$$g_x\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = g_{ij}(x, y) = \frac{\partial^2 F}{\partial y^i \partial y^j} \quad \left(F = \frac{1}{2}L^2\right)$$

(parameterized by $x \in M$ smoothly)

N^i_j: Cartan's non-linear connection

($\Longrightarrow$ Ehresmann connection)

$\Longrightarrow$

$$T(TM) = \mathcal{V}_{TM} \oplus \mathcal{H}_{TM}$$

Horizontal lift $\tilde{c}(t) = (c(t), y(t))$ of a smooth curve $c(t)$ in M

$\overset{\text{def}}{\Longleftrightarrow}$

$$\frac{dy^i(t)}{dt} + \sum N^i_j(c(t), y(t))\frac{dx^j}{dt} = 0 \quad \left(y^i(t) = \frac{dx^i}{dt}\right)$$

Parallel displacement $P_c : M_{c(0)} \to M_{c(1)}$ (diffeomorphism)

$\overset{\text{def}}{\Longleftrightarrow}$

$$P_c(y) = y(1) \quad (y = y(0))$$

$\Longrightarrow$ P_c preserves the norm for all $c(t)$, i.e.,

$$||y||_{c(0)} = ||P_c(y)||_{c(1)}$$

Landsberg space

$\Longleftrightarrow P_c : (M_{c(0)}, g_{c(0)}) \to (M_{c(1)}, g_{c(1)})$: **isometry** for all $c(t)$:

$$g_{c(0)}(Y, Z) = g_{c(1)}(P_{c*}(Y), P_{c*}(Z))$$

$\Longleftrightarrow \boxed{(\mathcal{L}_{X^H} g)^V = 0}$

Berwald space ($\subset$ Landsberg spaces)

$\Longleftrightarrow P_c : (M_{c(0)}, \|\cdot\|_{c(0)}) \to (M_{c(1)}, \|\cdot\|_{c(1)})$: **isometry** for all $c(t)$:

$$\|y - z\|_{c(0)} = \|P_c(y) - P_c(z)\|_{c(1)}$$

$\Longleftrightarrow P_c : M_{c(0)} \to M_{c(1)}$: **linear** for all $c(t)$

$\left(\Longrightarrow \quad (\mathcal{L}_{X^H} g)^V = 0, \text{ i.e., Berwals spaces} \subset \text{Landsberg}\right)$

$\Longrightarrow$ there exists a Riemannian metric g on M such that

$$N^i_j(x, y) = \sum \left\{ {i \atop jk} \right\} y^k \quad \text{(cf. [Sz])}$$

Locally Minkowski space ($\subset$ Berwald spaces)

$\Longleftrightarrow$ Berwald and the associated Riemannian metric g is **flat**

$\Longleftrightarrow$ there eists a coordinate system $(x^1, \cdots, x^n)$ on M such that

$$N^i_j(x, y) = 0 \quad (\Longleftrightarrow L = L(y))$$