Rigidity issues in Finsler geometry

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0. Finsler spaces = Differentiable length spaces

Let M be a differentiable manifold and $\gamma : [a,b] \to M$ is a curve. The length of γ is

$$\int_a^b ||\dot{\gamma}(t))||dt.$$

Finsler geometry is actually the geometry of a simple integral and is as old as the calculus of variations.

Hilbert: Paris address of 1900 devoted Problem 23 to the variational calculus of an invariant integral and its geometrical .

Relationships between topology and curvature

- Gauss-Bonnet-Chern Theorem; Bao-Chern '96, Shen '98
- Cartan-Hadamard Theorem, Bonnet-Myers Theorem, Synge's Theorem, ; Auslander '555
- The First Comparison Theorem of Rauch; Bao-Chern '93
- Bishop-Gromov Volume Comparison Theorem; Shen '97
- Sphere Theorem; Dazord '68, Kern '71, Rademacher '02

An Alexandrov space is a length space with (sectional) curvature bounded below or above in the distance comparison sense.

$K \geq 0$; the concavity of distance functions; Cheeger-Gromoll Theorem, Yamaguchi Fibration Theorem, Perelman and Cheeger-ColdingStability Theorems

$K \leq 0$; the convexity of distance functions; Mostow-Kleiner Rigidity Theorem, Besson-Courtois-Gallot Theorem.

Remark Alexandrov and Finsler geometry

(1) Fisnler spaces whose curvature is bounded below or above in the distance comparison(Alexandrov) sense are automatically Riemannian.

(2) If in a Minkowski space(normed space) an angle in the sense of Alexandrov exists for any geodesic rays, then the space is Euclidian.

Serious problem arises here.

1. Motivation and Historical Remarks

Boundary Rigidity Problems: Let $(M, \partial M, g_0)$ and $(M, \partial M, g)$ be compact Riemannian manifolds with boundary. The metric g on induces a distance function d from $\partial M \times \partial M$ to $\mathbf{R}$. For what $(M, \partial M, g_0)$ is it true that any $(M, \partial M, g)$ with $d_0 = d$ must have g_0 isometric to g?

[Michel '81, Gromov '83, BCG '96] Any compact subdomain with smooth boundary of $\mathbf{R^n}$, hyperbolic space H^n, or the open hemisphere of S^n are boundary rigid.

[Arcostanzo '94]* The boundary rigidity problem has a negative answer in the Finsler case; [Yim-K '00]

[Howard '96] compact subdomain on Lorentzian surfaces.

(Finsler Version) Hopf Conjecture:

[Burago, Ivanov '94] Any Riemannian metric without conjugate points on torus is flat.

[Busemann '55] There are metrizations of the torus without conjugate points for which the universal covering space is not Minkowskian. For example,

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$
$$+|7y_1 + \sin(2\pi y_1) - 7y_2 - \sin(2\pi y_2)|,$$

i.e.,

$$F_d((x, y); (u, v)) = \sqrt{u^2 + v^2} + |7v + 2\pi v \cos(2\pi y)|.$$

[Dazord '71] If a two-dimensional torus is a Landsberg and has no conjugate points, then the space is isometric to a flat Finsler torus.

(sketch of proof) Gauss-Bonnet Theorem

$$\int_{SM} Ric\, v \cdot \sigma_0 = 4\pi^2 \chi(M),$$

where σ_0 is the volume of the indicatrix.

If the space is compact and no conjugate points then the above integral ≥ 0 and equality holds iff the flag curvature vanishes.

[Burago's problem '01] If a Finsler torus has no conjugate points, is it alf-equivalent to a flat one?

alf-equivalent: there exist $F : S^*M \to S^*N$ and $f : S^*M \to R$ such that $F^*\omega_N = \omega_M + df$.

Burago-Ivanov's three problems:

(1) Minimality of flats in Banach spaces: Does a ball in an n-dimensional affine subspace of a Banach space minimize volume among all n-dimensional surfaces with same boundary?

(2) Finsler filling volume problem: Does its flat Finsler metric have the least volume among all Finsler metric whose distances between boundary points majorize those of the Banach distance function?

(3) Finsler volume growth problem: Does the volume of balls in the universal cover of a Finsler torus asymptotically grows at lest as fast as that in a Banach space?

[Burago-Ivanov '01] For every n, the three problems are equivalent.

[Ivanov '01] (1) is true in $n = 2$.

2. Preliminaries

Let M be an n-dimensional smooth manifold and TM denote its tangent bundle. A *Finsler structure* on a manifold M is a map $F : TM \to [0, \infty)$ which has the following properties:

(i) F is smooth on $\widetilde{TM} := TM \setminus \{0\}$;

(ii) $F(ty) = |t|F(y), t \in \mathbf{R}, \ \ y \in T_xM$;

(iii) $\frac{1}{2}F^2$ is strongly convex, i.e., $g_{ij}(x,y) := \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}(x,y)$ is positive definite for all $(x,y) \in \widetilde{TM}$. Here $F(x,y) := F(y^i \frac{\partial}{\partial x^i}|_x)$ and $y = (y^i)$.

(i) F^2 is C^2 on all TM if and only if F is the norm of a Riemannian metric.

(ii) Homogeneous of degree one in y and symmetric condition: The Hilbert form $\omega = F_{y_i} dx^i$ is essentially Hilbert's invariant integral in the calculus of variations; the length of curve is $\int_a^b w$.

(iii) Regularity Hypothesis:

[Chern '48] The Hilbert form ω is a global one form on SM and define a contact structure $\omega \wedge d\omega \neq 0$ on SM.

[Mercuri '77] The critical point theory for the closed geodesics problem; Klingenberg

Triangle Inequality

Minkowski space

F is called Riemannian if $g_{ij}(x) = g_{ij}(x, y)$ are independent of y.

F is called locally Minkowskian if $g_{ij}(y) = g_{ij}(x, y)$ are independent of x.

TFAE

(1) Locally Minkowskian space

(2) $R = P = 0$; under Chern connection

(3) [Zadeh '88] $R = 0$ and $\sup_{x \in B(r)} |A| = \sup_{x \in B(r)} |\nabla_{\text{vert}} A|$
$= o(r)$ as $r \to \infty$

Spaces of constant curvature

All of simply connected complete Riemannian manifolds with constant sectional curvature K are isometric to Eucldean $K = 0$, sphere $K > 0$, or hyperbolic spaces $K < 0$; Cartan, Hopf.

<u>Zadeh Theorem '88</u>

Let (M, F) be a compact Finsler manifold with constant flag curvature R.

(1) If $R < 0$, then F is Riemannian.

(2) If $R = 0$, then F is locally Minkowskian.

The non-Riemannian $R = 1$ examples constructed by Bryant '97 and Bao-Shen '01, Yim-K '01, Foulon '02 .

Negatively curved manifolds; Foulon '97, Boland-Newberger '01, Inkang Kim '01, Colbois-Verovic '02

Mean Tangent Curvature

Let F_x denote the restriction of F onto T_xM. When F is Riemannian, (T_xM, F_x) are all isometric to the Euclidean spaces $\mathbf{R^n}$. For a general Finsler metric F, however, (T_xM, F_x) may not be isometric to each other.

Let $\{e_i\}_{i=1}^n$ be a local basis for TM. Put $B_x(1) := \{y = (y^i) : F(y^i e_i) \leq 1\}$. Let $\mathbf{B}^n(1)$ denotes the standard unit ball in $\mathbf{R}^n$.

Define the *mean distortion* $\rho : \widetilde{TM} \to (0, \infty)$ by

$$\rho(v) := \frac{\mathrm{vol}_0(\mathbf{B}^n(1))}{\mathrm{vol}_0(B_x(1))} \frac{1}{\sqrt{\det(g_{ij}^v)}} := \frac{\sigma(x)}{\sqrt{\det(g_{ij}^v)}}.$$

The *mean tangent curvature* $H : \widetilde{TM} \to \mathbf{R}$ is defined by

$$H(v) := \frac{d}{dt}\left\{\ln \rho(\dot{\gamma_v}(t))\right\}|_{t=0}.$$

Geometric Meaning of Mean Tangent Curvature

(i) The mean tangent curvature $H(v)$ measures the *average change* of (T_xM, F_x) in the direction $v \in T_xM$.

(ii) $H = 0$ for Finsler manifolds modeled on a single Minkowski space. In particular, $H = 0$ for Berwald spaces. Locally Minkowski spaces and Riemann spaces are all Berwald spaces .

(iii) For a local smooth distance function ρ, $\Delta\rho = \overline{\Delta}\rho + H(\nabla\rho)$, where $\Delta\rho$ and $\overline{\Delta}\rho$ denote the Laplacian of ρ with respect to F and $g^{\nabla\rho}$, respectively. $\Rightarrow$ Bishop-Gromov volume comparison with an extra condition .

$$V_{\lambda,\mu}(r) := \mathrm{vol}_0(\mathbf{S}^{n-1}) \int_0^r e^{\mu t} s_\lambda(t)^{n-1} dt,$$

where $s_\lambda(t)$ denotes the unique solution to $y^{''} + \lambda y = 0$ with $y(0) = 0, y'(0) = 1$.

Bishop-Gromov Volume Comparison Theorem [Shen '97]

Let (M, F) be a complete Finsler manifold.

1. If $\mathrm{Ric}_M \geq (n-1)\lambda$, $|H| \leq \mu$, then for all $x \in M$ and for any $0 < r < R$ we have

$$\frac{\mathrm{vol}_F(B(x,r))}{V_{\lambda,\mu}(r)} \geq \frac{\mathrm{vol}_F(B(x,R))}{V_{\lambda,\mu}(R)}.$$

2. If (M, F) has flag curvature bounded above by λ and $|H| \leq \mu$, then for every $x \in M$ and $0 < r < R \leq \min\{\mathrm{inj}_x, \pi/\sqrt{\lambda}(= \infty \text{ if } \lambda \leq 0)\}$ we have

$$\frac{\mathrm{vol}_F(B(x,r))}{V_{\lambda,\mu}(r)} \leq \frac{\mathrm{vol}_F(B(x,R))}{V_{\lambda,\mu}(R)}.$$

Furthermore, we have equality in the above statements if and only if any Jacobi field $J_u(t)$ along γ_v has the following form $J_u(t) = s_\lambda(t) \cdot u(t)$, where $u = u(t)$ is a parallel vector field along γ_v.

3. Main results

[Shen '98] Is a reversible Finsler metric with $R = 1$ and $H = 0$ Riemannian metric? Yes, [Yim and K '01]

Duran Theorem. Let (M, F) be an n-dimensional compact reversible Finsler manifold. Then we have

$$V(SM) \leq \alpha(n-1) \cdot vol_F(M),$$

with equality if and only if (M, F) is a Riemannian metric.

(sketch of proof) The universal covering $\overline{M}$ of M is an n-dimensional Finsler manifold with flag curvature $R = 1$, then every geodesic is closed with same length 2π and $\overline{M}$ is a diffeomorphic sphere. By Weinstein and Yang's result, the symplectic volume of $\overline{M}$, $V(S\overline{M})$ is equal to $V(SS^n)$.

Since $R = 1$ and $H = 0$, by Bishop-Gromov Volume Comparison

Theorem, we have $vol_F(\overline{M}) = \alpha(n)$.

$$\begin{aligned} V(SS^n) &= \alpha(n-1)\cdot\alpha(n) \\ &= \alpha(n-1)\cdot vol_F(\overline{M}) \\ &\geq V(S\overline{M}) = V(SS^n) \end{aligned}$$

and hence we obtain $V(S\overline{M}) = \alpha(n-1)\cdot vol_F(\overline{M})$. Then by the equality case of Duran Theorem, we conclude that $(\overline{M}, F)$ is a Riemannian manifold, and hence F is a Riemannian metric.

Corollary 1

Let (M, F) be an n-dimensional reversible Finsler manifold with Ricci curvature bounded blew by $(n-1)$ and mean tangent curvature $H = 0$. If the diameter of M is equal to π, then M is isometric to the standard Riemannian sphere S^n of constant sectional curvature one.

Corollary 2

Let (M, F) be an n-dimensional reversible Finsler manifold with scalar curvature bounded blew by $n(n-1)$ and mean tangent curvature $H = 0$. If the conjugate radius of M is equal to π, then M is isometric to the standard Riemannian sphere S^n of constant sectional curvature one.

Santaló's Formula

Let dV denote the symplectic volume form of the Sasaki metric on $\widetilde{TM}$,

$$\begin{aligned} dV(v) &= \sqrt{\det(g_{ij}(v))}dx \wedge \sqrt{\det(g_{ij}(v))}dy \\ &= (\rho(v))^{-1} dV_x(v) dv(x) \\ &= (-1)^{\frac{n(n-1)}{2}} \frac{1}{n!} \underbrace{d\omega \wedge \cdots \wedge d\omega}_{n\text{-times}}. \end{aligned}$$

We recall that the geodesic flow ϕ_t preserves $d\omega$ and $\frac{d}{dt}\{(\phi_t)^*(i^*(dV))\} = 0$ (Liouville-Dazord's theorem '69). A proof of Santaló's formula in the Finsler case which only uses the fact the geodesic flow is volume preserving .

Let $(\Omega, \partial\Omega)$ be a compact domain in M and for any $v \in S\Omega$,

$$\tau(v) := \sup\{\tau > 0 : \gamma_v(t) \in \Omega, t \in (o, \tau)\},$$

$$S^+\partial\Omega := \{v \in S\Omega : g^\nu(v, \nu) > 0 \text{ on } x = \pi(v) \in \partial\Omega\}$$

with measure $(\rho(v))^{-1} dV_x(v) da(x)$, where da denotes the induced measure on $\partial\Omega$.

Santaló's formula

For all integrable function f on $S\Omega$ we have

$$\int_{S\Omega} f dV = \int_{S^+\partial\Omega} \left\{ \int_0^{\tau(v)} f(\phi_t(v))(\rho(\phi_t(v)))^{-1} g^\nu(v, \nu) dt \right\} dV_x(v) da(x).$$

To understand the geometric meaning of the mean tangent curvature, we consider another volume form $d\mu$ on $\widetilde{TM}$ defined by

$$d\mu(v) := \sigma^2(x)dx \wedge dy = \rho^2(v)dV(v).$$

<u>Proposition, Shen '99</u>

For the volume form $d\mu$ on SM defined above, we have

$$\frac{d}{dt}\left\{(\phi_t)^*(d\mu(v))\right\} = 2H(\phi_t(v))d\mu(\phi_t(v)).$$

Hence if $H = 0$, then the measure $d\mu$ invariant under the geodesic flow ϕ_t.

In the case of Finsler metric with vanishing mean tangent curvature, we obtain a simpler form of Santaló's formula with respect to $d\mu$.

Santaló's formula

Let τ, $S\Omega$, f, $d\mu$ be as above. If the mean tangent curvature vanishes on Ω, then we obtain

$$\int_{S\Omega} f\, d\mu = \int_{S^+\partial\Omega} \left\{ \int_0^{\tau(v)} f(\phi_t(v)) g^\nu(v,\nu) dt \right\} d\mu_x(v) da(x).$$

Let Ω be a compact differentiable manifold. Then the Riemannian relation $\mathrm{vol}(S\Omega) = \alpha(n-1) \cdot \mathrm{vol}(\Omega)$ breaks down in the Finsler case. However, the volume of the unit tangent bundle $S\Omega$ with respect to $d\mu$ is given by

$$\mathrm{vol}_\mu(S\Omega) = \int_\Omega \left\{ \int_{S_x\Omega} d\mu_x(v) \right\} dv(x) = \alpha(n-1) \cdot \mathrm{vol}_F(\Omega).$$

Thus we have the following.

Lemma. Let $\Omega, S\Omega$ and $d\mu$ be as above. Then we have

$$\mathrm{vol}_\mu(S\Omega) = \alpha(n-1) \cdot \mathrm{vol}_F(\Omega).$$

Theorem [Yim and K '00]

Let (V, F_0) be a Minkowski space and F any other Finsler metric with vanishing mean tangent curvature on V that has no conjugate points and agrees F_0 outside a compact set K. Then (V, F) is isometric to (V, F_0).

(sketch of proof) First we adapt Croke's argument to obtain $d_F = d_{F_0}$ on $\partial K \times \partial K$ in the Finslerian setting.

We may assume that the compact set K is contained in the interior of the cube $C = \{(x_1, x_2, \cdots, x_n) : |x_i| \leq R, i = 1, 2, \cdots, n\}$ for some R.

By Santaló's formula, we obtain $\mathrm{vol}_F(C) = \mathrm{vol}_{F_0}(C) = (2R)^n$.

Let N be the face of C given by $x_n = -R$ and for each $x \in N$ let $\gamma_v(s) : [0, 2R] \to V$ be the geodesic on F from $x = \gamma_v(0)$

perpendicular to N, i.e., $v = \nu$. Then

$$\mathrm{vol}_F(C) = \int_N \int_0^{2R} \det(B_v(s))\,ds\,da.$$

For any positive function $\det(B_v(s))$ the Hölder inequality implies

$$\begin{aligned}\mathrm{vol}_F(C) &\geq \int_N (2R)^{(n+1)/2}(2R)^{-(n-1)/2}\,da \\ &= 2R\int_N da = 2R(2R)^{n-1} = (2R)^n\end{aligned}$$

with equality holding if and only if $B_v(s)$ is the identity matrix for all v and s. But we know that equality holds, hence $B_v(s)$ is the identity. Thus we have the flag curvature along γ_v is constant zero, and by the same as the technical proof of Zadeh, we conclude that the Finsler metric F is the Minkowski metric.

As a direct consequence of above theorem and the Cartan-Hadamard theorem, we have:

Corollary

Any complete Finsler metric with nonpositive flag curvature on a vector space which has vanishing mean tangent curvature and is isometric to the Minkowski metric outside a compact set must be isometric to the Minkowski metric.

Remark

Croke used the convexity of distance functions to extended his result to simply connected manifolds. However a Finsler manifold does not have the distance of convexity, we can not directly the above theorem as in Riemannian case.

4. A summary of the main points

One of the important problem in Finsler geometry is to classify all Finsler spaces of constant flag curvature. In this talk we are understood the geometric meaning of the mean tangent curvature and shown that a reversible Finsler metric with positive constant flag curvature and vanishing mean tangent curvature must be Riemannian. For a Euclidean space or a Minkowski space, we change the metric in a compact subset and show that the resulting Finsler manifold is isometric to the original standard space under certain conditions. We assume that the mean tangent curvature vanishes and the metric satisfies some curvature conditions or have no conjugate points.