G. Zuckerman

A generalization of Harish-Chandra discrete series to Lie superalgebras

1) $(\mathfrak{g}, \mathfrak{k})$-modules, or "Beyond category $\mathcal{O}$"
2) Harish-Chandra's discrete series of particular $(\mathfrak{g}, \mathfrak{k})$-modules
3) Blattner's formula and cohomological induction
4) Super analogs

$\mathfrak{g}$ is a fin. dim Lie superalg / $\mathbb{C}$

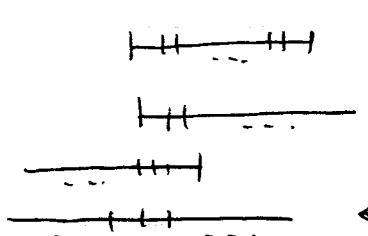
$\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$

Let $\mathfrak{k}$ be a subalgebra of $\mathfrak{g}_{\bar{0}}$ such that $\mathfrak{k}$ is reductive in $\mathfrak{g}$.

<u>Def</u>. A $\mathfrak{g}$-module M is a $(\mathfrak{g}, \mathfrak{k})$-module if it decomposes into fin. dim. irred. $\mathfrak{k}$-submodule

Kostant
Harish-Chandra
Gelfand

Ex $\mathfrak{g} = sl(2)$, $\mathfrak{k} = so(2)$



fin

} 1-sided

} determined by restriction to $\mathfrak{k}$.

← need additional parameter (e. values of Casimir op)

Basic questions: assume M is an irreducible $(\mathfrak{g}, \mathfrak{k})$-module.

1) When does M have highest weight vector relative to some Borel subalg $\mathfrak{b} \subseteq \mathfrak{g}$?
2) When is the equiv. class of M determined by $\mathrm{Res}_{\mathfrak{k}} M$?
3) When are the $\mathfrak{k}$-multiplicities finite?

Study a general semisimple Lie alg $\mathfrak{g}$,
assume $\mathfrak{k} = \mathfrak{g}^{\theta}$, $\theta \in \mathrm{Aut}(\mathfrak{g})$
$\theta^2 = 1$, $\theta \neq 1$.

Harish-Chandra ~1950

Dixmier, Enveloping Algebras (1976), book.

Further restriction: assume $\mathrm{rank}\, \mathfrak{k} = \mathrm{rank}\, \mathfrak{g}$
Let $\mathfrak{t}$ be a Cartan subalg in $\mathfrak{k}$.
($\mathfrak{t}$ is also a Cartan subalg in $\mathfrak{g}$).

Harish-Chandra discrete series is a family of irred $(\mathfrak{g}, \mathfrak{k})$-modules parametized by $\lambda \in \mathfrak{t}'$ subject to
a) λ is regular (relative to $\mathfrak{g}$)
b) λ is dominant integral (relative to $\mathfrak{k}$).

$\lambda \longmapsto D_\lambda$: this module is rigid in the sense that $\mathrm{res}_{\mathfrak{k}} D_\lambda$ <u>determines</u> D_λ as a $\mathfrak{g}$-module.

$D_\lambda \cong D_\mu \Leftrightarrow \lambda = \mu$.

Example. $\mathfrak{g} = sl(3)$, $\mathfrak{k} = gl(2)$. $\left[\begin{array}{c|c} /// & 0 \\ \hline 0 & /// \end{array}\right]$ $\mathfrak{t}$ = diagonal

roots of $\mathfrak{t}$ in $\mathfrak{g}$: $\pm(e_1 - e_2)$, $\pm(e_1 - e_3)$, $\pm(e_2 - e_3)$.

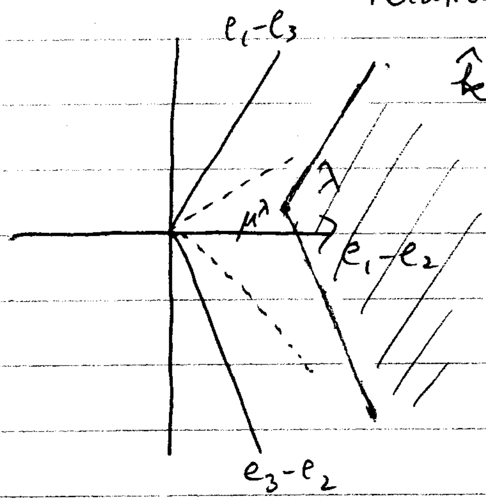
assume $e_1 - e_2$ is positive.

Choose λ regular and $(\lambda, e_1 - e_2) > 0$ and integral

Ex. Assume $(\lambda, e_1 - e_3) > 0$ and $(\lambda, e_3 - e_2) > 0$.

Δ_+ = positive roots : $e_1 - e_3$, $e_3 - e_2$, $e_1 - e_2$

In this case, D_λ is <u>not</u> a highest weight module relative to any Borel $\mathfrak{b} \supseteq \mathfrak{t}$.



$\hat{\mathfrak{k}}$ = set of classes of fin. dim irred. rep, indexed by right half plane

$(\mu, e_1 - e_2) \geq 0$.

$(\mu, e_1 - e_2) \in \mathbb{N}$.

shaded area is support (D_λ).

$\mathfrak{k}$-types have mult. <u>one</u> in

$\mathfrak{t}$-weights in D_λ have infinite multiplicities.

In general $\underbrace{\mathfrak{t} \subseteq \mathfrak{k}}_{\text{C.S.A.}} \subseteq \mathfrak{g}$, $\mathfrak{k} = \mathfrak{g}^\theta$

Choose $\Delta_{+,c}$ = positive roots for $\mathfrak{t}$ in $\mathfrak{k}$.

Choose $\lambda \in \mathfrak{t}'$ dominant integral for $\mathfrak{k}$, regular for $\mathfrak{g}$.

Let $\Delta_+ = \{ \alpha \in \Delta(\mathfrak{t}, \mathfrak{g}) \mid (\lambda, \alpha) > 0 \}$

$$\rho = \tfrac{1}{2}\langle \Delta_+ \rangle$$

$$\rho_c = \tfrac{1}{2}\langle \Delta_{+,c} \rangle$$

$$\mu^\lambda = \lambda + \rho - 2\rho_c$$

<u>Lemma</u>: μ^λ is $\mathfrak{k}$-dominant integral.

<u>Theorem</u> (Harish-Chandra, Vogan) ~1977

Given above data, there exists a unique irreducible $(\mathfrak{g}, \mathfrak{k})$-module D_λ such that μ^λ is a highest weight for a $\mathfrak{k}$-submodule of D_λ

and if μ is a " " " " " " " ",

$$\mu - \mu^\lambda = \langle A \rangle, \quad A \text{ is multiset of roots in } \underbrace{\Delta_+ \setminus \Delta_{+,c}}_{\Delta_{+,n}}$$

$$= \sum_{\alpha \in \Delta_{+,n}} n_\alpha \alpha, \quad n_\alpha \in \mathbb{N}.$$

<u>Theorem 2</u> (Hecht - Schmid)

If μ occurs as a $\mathfrak{k}$-highest weight in D_λ

mult of "Z^μ" is given by Blattner's formula:

irred. h. wt μ.

(Jeb Willenbring) $\operatorname{supp}_{\mathfrak{k}} D_\lambda$?

$$\sum_{w \in W(\mathfrak{t}, \mathfrak{k})} \det(w)\, P_{\Delta_n^+}\big(w(\mu + \rho_c) - (\mu^\lambda + \rho_c)\big).$$

$$P_{\Delta_n^+}(\nu) = \#\text{ ways you can write } \nu$$

$$\text{as } \sum_{\alpha \in \Delta_{+,n}} n_\alpha \alpha.$$

One construction of D_λ : cohomological induction

a) get a purely algebraic construction of D_λ.

b) get an immediate proof of Blattner's formula as mult. formula for D_λ.

b') get a proof that Blattner's alternating sum is <u>nonnegative</u> (as long as μ is $\mathfrak{k}$-dominant integral).

c) get an immediate generalization of D_λ theory to Lie superalgebras.

Super set-up:

$\mathfrak{g}$ a $\underbrace{\text{contragradient}}$ fin dim Lie superalg. "reductive"

$\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ (invariant nondegenerate even bilinear form)

<u>Ex</u>. $\mathfrak{g} = \mathfrak{gl}(m|n)$, $\mathfrak{osp}(m|2n)$.

$\mathfrak{k} \subseteq \mathfrak{g}_0$ maximal rank reductive subalg (not necessarily "symmetric").

$\mathfrak{t}$ is a C.S.A of $\mathfrak{k}$ $(\mathfrak{g}_{\bar{0}}, \mathfrak{g})$.

$\lambda \in \mathfrak{t}'$ regular relative to $\mathfrak{g}$.
λ is $\mathfrak{k}$-integral dominant.

<u>Theorem 3</u> By cohomological induction, <u>construct</u> a $(\mathfrak{g}, \mathfrak{k})$-module A_λ s.t.

a) μ^λ occurs as a highest $\mathfrak{k}$-wt. in A_λ.
$\mu^\lambda = \lambda + \rho - 2\rho_c$.

b) If μ occurs as a highest $\mathfrak{k}$-wt in A_λ,

$$\mu - \mu^\lambda = \sum_{\alpha \in \Delta_+ \setminus \Delta_{+,c}} n_\alpha \alpha \qquad \left(\begin{array}{l} n_\alpha \in \mathbb{N} \text{ if } \alpha \text{ is even} \\ n_\alpha \in \{0,1\} \text{ if } \alpha \text{ is odd.} \end{array} \right.$$

c) If μ occurs as a $\mathfrak{k}$-highest wt in A_λ
mult of Z^μ :

$$\sum_{w \in W(\mathfrak{t},\mathfrak{k})} \det(w)\, \mathcal{P}^{\text{"super"}}_{\Delta_{+,n}}\left(w(\mu+\rho_c) - (\mu^\lambda + \rho_c)\right)$$

$\mathcal{P}^{\text{super}}(\nu) = \#$ of ways $\nu = \sum n_\alpha \alpha$
$n_\alpha \in \{0,1\}$ if α is <u>odd</u>.

d). If $\mathfrak{g} = \mathfrak{g}_0$, $A_\lambda \cong D_\lambda$.

<u>Open problems</u> : 1) Prove A_λ is irreducible over $\mathfrak{g}$.
(probably okay for "very" regular λ)
(known for any $\mathfrak{k}$ in $\mathfrak{g}$ if $\mathfrak{g}$ is purely even)

2) Combinatorics of "super" Blattner formula

3). Is A_λ <u>determined</u> by $\mathrm{res}_{\mathfrak{k}} A_\lambda$?

e.g. $\mathfrak{g} = \mathfrak{gl}(4)$, $\mathfrak{k} = \mathfrak{z} + \mathfrak{sl}(2)_\alpha$. A_λ rigid?

"Superconformal group $SU(2,2/N)$"

4) $\mathfrak{g} = \mathfrak{gl}(4|4)$, $\mathfrak{k} = \mathfrak{gl}(2) \oplus \mathfrak{gl}(2) \oplus \mathfrak{gl}(4)$

Cohomological induction

$$\mathcal{C}(\mathfrak{g}, \mathfrak{t}) \underset{\text{forgetful}}{\overset{\Gamma_{\mathfrak{k},\mathfrak{t}}}{\rightleftarrows}} \mathcal{C}(\mathfrak{g}, \mathfrak{k})$$

$\Gamma_{\mathfrak{k},\mathfrak{t}} N$ = largest $(\mathfrak{g}, \mathfrak{k})$-submodule in N

$R^* \Gamma_{\mathfrak{k},\mathfrak{t}}$ = right derived functors of $\Gamma_{\mathfrak{k},\mathfrak{t}}$.

$R^j \Gamma_{\mathfrak{k},\mathfrak{t}} \neq 0$ iff $0 \leq j \leq \dim(\mathfrak{k}/\mathfrak{t})$.

$$A_\lambda \overset{\text{def}}{=} R^s \Gamma_{\mathfrak{k},\mathfrak{t}} \operatorname{ind}_{\bar{\mathfrak{b}}}^{\mathfrak{g}} \mathbb{C}_{\lambda+\rho}$$

$$\begin{pmatrix} s = \frac{1}{2} \dim(\mathfrak{k}/\mathfrak{t}) \\ = |\Delta_{+,c}| \end{pmatrix} \qquad \bar{\mathfrak{b}} = \mathfrak{t} + \sum_{(\lambda,\alpha)<0} \mathbb{C} X_\alpha$$

$$\rho = \frac{1}{2} \sum_{(\lambda,\alpha)>0} \alpha .$$

Case $\mathfrak{g} \supseteq \mathfrak{g}_0 = \mathfrak{k}$ <u>Santos</u>