New examples of non-Kähler homotopy types

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- joint work with T. Pantev and B.Toen.

- will describe new homotopy invariants of a topological space X, related to the action of $\pi_1(X, x)$ on the higher homotopy groups $\pi_i(X) \otimes \mathbb{C}$, $i > 1$.

- will explain the construction of a Hodge decomposition on the schematized homotopy type $(X \otimes \mathbb{C})^{\mathsf{sch}}$ of a smooth projective variety X.

- will construct some new examples of homotopy types which are not realizable as complex projective manifolds.

- will discuss generalizations, different realizations some other applications. For more details see math.AG/0107129.

1. Hodge theory and restrictions on Kähler homotopy types

All schemes will be over $\mathbb{C}$.

Problem 1. Find effective conditions that distinguish the homotopy types of compact Kähler manifolds among all homotopy types of finite CW complexes.

Remark: All known conditions are of Hodge theoretic nature:

- the existence of a Hodge structure on the cohomology, the fundamental group or the rational homotopy type of a projective manifold.

- the natural products, e.g. the cup product in cohomology, the Whitehead products on homotopy groups, etc., are morphisms of (mixed) Hodge structures.

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$\mathcal{P}$ = the class of homotopy types of all smooth projective varieties.

Sample Hodge-theoretic restrictions.
Suppose X = the homotopy type of a finite CW complex.

(i) Cohomological restrictions

• (Hodge, Weil) If $X \in \mathcal{P}$, then $b_{\mathrm{odd}}(X) \in 2\mathbb{Z}$.
Example: If $\pi_1(X)$ is free, then $X \notin \mathcal{P}$.

• (Johnson-Rees) If $H^1(\pi_1(X), \mathbb{Q}) \neq 0$ and $X \in \mathcal{P}$, then the cup product

$$H^1(\pi_1(X), \mathbb{Q})^{\otimes 2} \xrightarrow{\smile} H^2(\pi_1(X), \mathbb{Q})$$

has nonzero image.
Example: If $\pi_1(X)$ is a lattice in the 3×3 Heisenberg group, then $X \notin \mathcal{P}$.

(ii) Rational homotopy type restrictions

(Deligne, Goldman-Millson) Let G be an affine algebraic group over $\mathbb{C}$ and let $X \in \mathcal{P}$. Then Hom$(\pi_1(X), G)$ has at most quadratic singularities at points corresponding to semisimple representations.

(iii) Non-abelian Hodge theory restrictions

• (Gromov, Arapura-Bressler-Ramachandran) If $X \in \mathcal{P}$, then $\pi_1(X)$ has at most one end.

• (Simpson) If $X \in \mathcal{P}$, then any rigid representation of $\pi_1(X)$ is of Hodge type.
Example: If $\pi_1(X) = SL(n, \mathbb{Z})$ with $n \geq 3$, then $X \notin \mathcal{P}$.

Comments:

• Let W be a real algebraic group, G be its complexification and σ be the complex conjugation on G corresponding to W. A *Cartan involution* on G is an automorphism $C : G \to G$, s.t. $C^2 = 1$, $C \circ \sigma = \sigma \circ C =: \tau$ is such that G^τ is compact.

The group W is called a *group of Hodge type* if we can find a γ in the identity component of G so that Ad_γ is a Cartan involution.

• Using Tanaka duality we can reformulate the Hodge type restriction of Simpson by simply saying that $\pi_1(X)^{\mathrm{red}}$ is a group of Hodge type. In particular $\mathbb{C}^{\times\delta}$ acts on $\pi_1(X)^{\mathrm{red}}$ so that -1 is a Cartan involution. This action of $\mathbb{C}^{\times\delta}$ on $\pi_1(X)^{\mathrm{red}}$ is the Hodge decomposition on the fundamental group.

• (Simpson) If $X \in \mathcal{P}$ and

$$\Sigma^i_k(X) := \{\rho \in \mathrm{Hom}(\pi_1(X), \mathbb{C}^\times) | h^i(X, \rho) \geq k\},$$

then $\Sigma^i_k(X) \subset \mathrm{Hom}(\pi_1(X), \mathbb{C}^\times)$ is a translation of a subtorus by a torsion point.

Example: Put $\Gamma := \mathbb{Z}^a$ with a even and consider

- a torus $T = K(\mathbb{Z}^a, 1)$ with a base point $t \in T$;
- a space Y obtained from T by attaching m 2-spheres at t;
- a space X obtained by attaching ℓ 3-cells to Y with attaching maps $\alpha_i \in \pi_2(Y, t) = (\mathbb{Z}\Gamma)^m$, $i = 1, \ldots, \ell$.

View the collection $\{\alpha_i\}$ as a matrix $A \in \mathrm{Mat}_{\ell\times m}(\mathbb{C}\Gamma)$. Let $A(\rho) \in \mathrm{Mat}_{\ell\times m}(\mathbb{C})$ be the matrix obtained by evaluating A on a non-trivial algebra homomorphism $\rho : \mathbb{C}\Gamma \to \mathbb{C}$. Then

$$\Sigma^2_k(X) = \{\rho \in \mathrm{Hom}(\Gamma, \mathbb{C}^\times) |\ \mathrm{rk}(A(\rho)) \leq m - k\}.$$

By choosing A carefully we can get $\Sigma^2_k(X)$ which does not pass through any torsion point[a]. Thus $X \notin \mathcal{P}$.

[a]This characteristic of X will not be seen by the rational homotopy type.

Unifying theme of these examples: The existence of a Hodge decomposition on various homotopy invariants (e.g. cohomology, rational homotopy type, fundamental group, non-abelian cohomology, etc.) of a homotopy type $X \in \mathcal{P}$.

Goal: Given a $X \in \mathcal{P}$, find a uniform construction of a Hodge decomposition on all homotopy invariants of X, i.e. find a Hodge decomposition on X itself.

Interpretation: View the various Hodge decompositions on homotopy invariants as actions of the group $\mathbb{C}^\times$.

Example: The Hodge decomposition

$$H^n(X, \mathbb{C}) = \oplus H^{n-p}(\Omega^p_X)$$

on the cohomology of a projective variety X can be thought of as the representation of $\mathbb{C}^\times$ on $H^n(X, \mathbb{C})$ which has weight p on the piece $H^{n-p}(\Omega^p_X)$.

These results are obtained using only the first nonabelian cohomology - they show some potential so they are worth studying. Philosophically we would like to go in two directions:

1) Creating a "Huge Object" and then getting geometric information after localizing.

2) Studying all realizations of "Huge Object". Going back to Mathematics we need:

Caveat: To define a Hodge decomposition on a homotopy invariant of some $X \in \mathcal{P}$ we have to algebraize this invariant.

Thus before we even begin to look for a Hodge decomposition on a homotopy type $X \in \mathcal{P}$ we must resolve the following:

Problem 2. Find an algebraic geometric incarnation of a homotopy type X.

The answer to this problem turns out to be the notion of a schematic homotopy type due to B.Toen.

2. Schematization of homotopy types

The **Schematization Problem** was posed by Grothendieck in ‘Pursuing Stacks’. He asked if one can find a functorial assignment:

$$(CW \text{ complex } X) \longrightarrow \begin{pmatrix} \text{an algebraic object} \\ X \otimes S \to S \\ \text{defined for every scheme } S \end{pmatrix}$$

so that

$$H^{\bullet}(X, \mathcal{O}(S)) \cong H^{\bullet}(X \otimes S, \mathcal{O}).$$

for all S.

Such $X \otimes S$ will be called the schematic homotopy type of X/S

Remark: At a first glance, the motivic flavor of the schematization problem makes it tempting to require that $X \otimes S$ be a scheme over S.

This requirement seems to be unrealistic as soon as $\pi_i(X) \neq 0$ for some $i > 0$.

An efficient way of incorporating the non-triviality of $\pi_i(X)$ into the definition of $X \otimes S$ is to require that $X \otimes S$ be an algebraic stack of ∞-groupoids.

These are exactly the objects that Grothendieck is pursuing in 'Pursuing stacks'.

Warning: The naive schematization assignment will not work. For example, the assignment $X := K(\mathbb{Z}, n) \to K(\mathbb{G}_a, n)$ will not work because if $S = \mathrm{Spec}(\overline{\mathbb{F}}_p)$, then $H^2(X, \overline{\mathbb{F}}_p) = \overline{\mathbb{F}}_p$ but $H^2(K(\mathbb{G}_a, n), \mathcal{O}) = \mathrm{Hom}_{\mathrm{gp}}(\mathbb{G}_a, \mathbb{G}_a) \neq \mathbb{G}_a$ since it contains $\mathbb{G}_a$ plus the Frobenius.

We will use the Simpson-Tamsamani approach to higher stacks, in which *stacks of ∞-groupoids* are modeled on simplicial presheaves.

Concretely: for us a stack (of ∞-groupoids) is a pointed connected simplicial presheaf on the site $(\mathsf{Aff}/\mathbb{C})_{\mathsf{ffqc}}$ of affine complex schemes with the flat topology, considered up to homotopy.

Notation: • $\mathsf{SPr}_*(\mathbb{C})$ - the category of pointed simplicial presheaves together with its local projective model structure.

• $\mathsf{Ho}(\mathsf{SPr}_*(\mathbb{C}))$ - the corresponding homotopy category. The objects in $\mathsf{Ho}(\mathsf{SPr}_*(\mathbb{C}))$ will be called *stacks*.

Remark: This point of view is consistent with the traditional notions of a scheme or a stack. For example algebraic spaces (respectively ordinary stacks) are 0-truncated (respectively 1-truncated) objects $F \in \mathsf{Ho}(\mathsf{SPr}_*(\mathbb{C}))$.

Comment: The local projective model structure on $\mathsf{SPr}_*(\mathbb{C})$ is the presheaf version of the Braun-Gersten model structure and is built (See B. Blander, paper 0462 in the K-theory archive) in two steps. First one has the following important

Theorem [Hirschhorn] $\mathsf{SPr}_*(\mathbb{C})$ is a proper simplicial cellular model category if we define weak equivalences and fibrations to be the section wise weak equivalences and fibrations and the cofibrations to be all maps with the left lifting property w.r.t. the trivial fibrations.

Remark: • The model structure described in the previous theorem is called the *projective model structure* on simplicial presheaves.
• The projective cofibrations turn out to be exactly the retracts of transfinite compositions of pushouts along maps $\partial\Delta^n \otimes h_X \to \Delta^n \otimes h_X$, where $X \in (\mathsf{Aff}/\mathbb{C})$.

The second step is to localize this model structure by the local weak equivalences:

Theorem [Blander] $\mathrm{SPr}_*(\mathbb{C})$ is a proper simplicial cellular model category if we define weak equivalences to be local weak equivalences, cofibrations to be the projective cofibrations and fibrations to be all maps with the right lifting property w.r.t. acyclic cofibrations.

Remark [Dugger]: The local projective model structure can be obtained from the projective one if we Bousfield localize along the class of all maps $h_{U_*} \to h_X$ where, $U_* \to X$ is a hypercover in the category of affine schemes. Equivalently F is fibrant in the local projective model structure iff F is fibrant in the projective model structure and F satisfies descent for hypercovers.

With this in mind we have the following

Definition: A pointed connected stack $F \in \mathrm{Ho}(\mathsf{SPr}_*(\mathbb{C}))$ is called a *schematic homotopy type*, if the sheaves of groups $\pi_i(F,*)$ are represented by affine group schemes for all $i > 0$ and if these group schemes are unipotent for $i > 1$.

Define now the schematization $(X \otimes \mathbb{C})^{\mathsf{sch}}$ as the universal schematic homotopy type to which X maps. The universal property implies that for any pointed schematic homotopy type F, one has an equivalence of mapping spaces

$$\mathbb{R}\underline{\mathsf{Hom}}(X,F) \simeq \mathbb{R}\underline{\mathsf{Hom}}((X \otimes \mathbb{C})^{\mathsf{sch}}, F).$$

Remark: Here $\underline{Hom}(F, G)$ denotes the simplicial presheaf of homomorphisms between two simplicial presheaves F, G and $\mathbb{R}\underline{Hom}(F, G)$ is its derived version, i.e.

$$\mathbb{R}\underline{Hom}(F, G) := \underline{Hom}('F, G') \in \mathsf{Ho}(\mathsf{SPr}_*(\mathbb{C})),$$

where $'F \to F$ is a cofibrant replacement and $G \to G'$ is a fibrant replacement.

We can now state Toen's schematization theorem:

Theorem (Toen) Let (X, x) be any pointed connected simplicial set. Then the schematization $(X \otimes \mathbb{C})^{\mathsf{sch}}$ exists and moreover:

(i) The affine group scheme $\pi_1((X \otimes \mathbb{C})^{\mathsf{sch}}, x)$ is naturally isomorphic to the pro-algebraic completion of $\pi_1(X, x)$ over $\mathbb{C}$.

(ii) There is a natural isomorphism

$$H^\bullet(X, \mathbb{C}) \cong H^\bullet((X \otimes \mathbb{C})^{\mathsf{sch}}, \mathbb{G}_a).$$

(iii) if X is simply connected and of finite type, then $\pi_i((X \otimes \mathbb{C})^{\mathsf{sch}}, x)$ is naturally isomorphic to the pro-unipotent completion of $\pi_i(X, x)$ over $\mathbb{C}$, i.e.

$$\pi_i((X \otimes \mathbb{C})^{\mathsf{sch}}, x) = \pi_i(X, x) \otimes_{\mathbb{Z}} \mathbb{G}_a.$$

3. The Hodge decomposition on $(X \otimes \mathbb{C})^{\mathsf{sch}}$

The construction of the Hodge filtration in rational homotopy theory, utilizes the fact that the cochain algebra $C^{\bullet}_{dR}(X, \mathbb{C})$ of a simply connected projective X can be used as an algebraic model of the complex homotopy type of X. In particular, the Hodge decomposition on the level of differential forms yields a Hodge decomposition on the commutative dga $C^{\bullet}(X, \mathbb{C})$.

Before we can apply the same reasoning to the general setting we need to resolve the following

Problem 3. Find an algebraic model for $(X \otimes \mathbb{C})^{\mathsf{sch}}$.

We describe a construction of such a model, inspired by a remark of P.Deligne concerning the homotopy groups $\pi_i((X \otimes \mathbb{C})^{\mathsf{sch}}, x)$.

The construction is intimately related to the way one does algebraic geometry in the category of stacks Ho(SPr($\mathbb{C}$)).

We need the notion of an *affine stack* which should be thought of as a derived version of the notion of an affine scheme.

Given a cdga A one defines a simplicial presheaf $\mathsf{Spec}(A) \in \mathsf{SPr}(\mathbb{C})$ by setting

$$\begin{array}{rccc} \mathsf{Spec}(A) \ : & (\mathsf{Aff}\,/\mathbb{C}) & \longrightarrow & (\mathsf{SSet}) \\ & \mathsf{Spec}(B) & \longmapsto & \mathsf{Hom}_{\mathsf{dga}}(A, B) \end{array}$$

Furthermore, for any cdga A one defines a stack $\mathbb{R}\,\mathsf{Spec}(A) \in \mathsf{Ho}(\mathsf{SPr}(\mathbb{C}))$ by setting

$$\mathbb{R}\,\mathsf{Spec}(A) := \mathsf{Spec}(A'),$$

where A' is a cofibrant replacement of A in the cmc of cdga.

Definition A stack $F \in \mathsf{Ho}(\mathsf{SPr}(\mathbb{C}))$ is called *affine* if there exists a commutative dga A concentrated in non-negative degrees so that

$$F \cong \mathbb{R}\,\mathsf{Spec}(A)$$

General fact: Any simply connected schematic homotopy type is an affine stack.

Comments: • A cdga A is *cofibrant* if it is built by a transfinite sequence of pushouts of free dga.

• A stack F is affine iff the dga $\mathbb{L}\mathcal{O}(F)$ of cochains of F with coefficients in $\mathbb{G}_a$ is ‘small’ and the natural map $F \to \mathbb{R}\,\mathrm{Spec}(\mathbb{L}\mathcal{O}(F))$ is an equivalence of simplicial presheaves.

Using the **General fact** we get the desired algebraic model of $(X \otimes \mathbb{C})^{\mathsf{sch}}$:

Consider the natural map

$$(X \otimes \mathbb{C})^{\mathsf{sch}} \xrightarrow{p} K(\pi_1((X \otimes \mathbb{C})^{\mathsf{sch}}), 1) = K(\pi_1(X)^{\mathsf{alg}}, 1),$$

and let $\widetilde{X \otimes \mathbb{C}}$ be the homotopy fiber of p. Then $\widetilde{X \otimes \mathbb{C}}$ is a simply connected schematic homotopy type and so one can find a cdga A, so that $\widetilde{X \otimes \mathbb{C}} \cong \mathbb{R}\,\mathsf{Spec}(A)$.

Conclusion: $\pi_1(X)^{\mathsf{alg}}$ acts on $\mathbb{R}\,\mathsf{Spec}(A)$ and $(X \otimes \mathbb{C})^{\mathsf{sch}} \cong [\mathbb{R}\,\mathsf{Spec}(A)/\pi_1(X)^{\mathsf{alg}}]$.

To make this algebraic model explicit we need to calculate A.

The answer is given by

Theorem (-,Pantev,Toen) *Let X be a pointed connected homotopy type and let G be the pro-reductive completion of $\pi_1(X)$ over $\mathbb{C}$. View the algebra of functions $\mathcal{O}(G)$ together with its $\pi_1(X)$ action as a local system of algebras on X and let $C^\bullet(X, \mathcal{O}(G))$ be the algebra of cochains on X with coefficients in $\mathcal{O}(G)$. Then*

$$(X \otimes \mathbb{C})^{\mathsf{sch}} \cong [\mathbb{R}\,\mathsf{Spec}(C^\bullet(X, \mathcal{O}(G)))/G].$$

Remark: The theorem remains true if we take G to be the pro-algebraic completion of $\pi_1(X)$, rather than the pro-reductive one.

A great advantage of the model

$$[\mathbb{R}\,\mathsf{Spec}(C^\bullet(X,\mathcal{O}(G)))/G]$$

is that it is related to the geometry of X.

If X is a smooth projective variety $C^\bullet(X,\mathcal{O}(G))$ can be computed from the de Rham complex of the local system $\mathcal{O}(G)$.

Using Simpson's non-abelian Hodge correspondence one can then relate this dga to the Čech cochain algebra computing the cohomology of certain Higgs bundles on X.

Combined with the rescaling action of $\mathbb{C}^\times$ on Higgs bundles this yields the Hodge decomposition on the schematic homotopy type, i.e. provides an action of the group $\mathbb{C}^{\times\delta}$ on the stack $(X\otimes\mathbb{C})^{\mathsf{sch}}$.

This result is summarized in the following:

Theorem (-, Pantev, Toen) *Let X be a pointed projective manifold. There exists an action of $\mathbb{C}^{\times\delta}$ on $(X \otimes \mathbb{C})^{\text{sch}}$ so that:*

- *The induced action of $\mathbb{C}^{\times\delta}$ on the cohomology groups $H^\bullet((X \otimes \mathbb{C})^{\text{sch}}, \mathbb{G}_a) = H^\bullet(X, \mathbb{C})$ is compatible with the Hodge decomposition.*
- *The induced action of $\mathbb{C}^{\times\delta}$ on $\pi_1(X)^{\text{red}}$ coincides with the one defined by Simpson.*
- *If X is simply connected, then the induced action of $\mathbb{C}^{\times\delta}$ on*

$$\pi_i((X \otimes \mathbb{C})^{\text{sch}})(\mathbb{C}) \cong \pi_i(X) \otimes \mathbb{C}$$

  coincides with the Hodge decomposition defined by Deligne-Griffiths-Morgan-Sullivan.
- *If $R_n := \text{Hom}(\pi_1(X), GL_n(\mathbb{C}))/GL_n(\mathbb{C})$, then the induced action of $\mathbb{C}^{\times}$ on R_n is continuous in the analytic topology.*

4. The support invariants and restrictions on homotopy types

For any pointed connected homotopy type X we can use the schematization $(X \otimes \mathbb{C})^{\mathrm{sch}}$ to construct a new homotopy invariant of X which is related in a subtle way to the action of $\pi_1(X)$ on the higher homotopy groups of X.

Let F be a pointed schematic homotopy type. By definition $\pi_1(F, *)$ is an affine group scheme and one can show that $\pi_i(F, *)$ are all abelian unipotent group schemes for $i > 0$.

Let $\pi_1(F, *)^{\mathrm{red}}$ be the maximal reductive quotient of $\pi_1(F, *)$ considered as a subgroup of $\pi_1(F, *)$ via the Levi decomposition.

Since $\pi_i(F, *)$ is a linearly compact vector space and $\pi_1(F, *)^{\mathrm{red}}$ is an affine reductive group scheme acting on it we get a decomposition as a (possibly infinite) product

$$\pi_i(F, *) = \prod_{\rho \in R} \pi_i(F, *)^{\rho}.$$

Comments:

• R denotes the set of isomorphism classes of finite dimensional simple linear representations of $\pi_1(F,*)$ and $\pi_i(F,*)^\rho$ is a (possibly infinite) product of representations of class ρ.

• Using the fact that the Levi decomposition is unique up to an inner automorphism one can check that the set

$$\{\rho \in R | \pi_i(F,*)^\rho \neq 0\}$$

is well defined and independent of the choice of the embedding $\pi_1(F)^{\mathrm{red}} \subset \pi_1(F)$.

Definition Let F be a pointed schematic homotopy type. The subset

$$\mathrm{Supp}(\pi_i(F,*)) := \{\rho \in R | \pi_i(F,*)^\rho \neq 0\} \subset R$$

is called the support of $\pi_i(F,*)$ for every $i > 1$.

The naturality of the construction of the Hodge decomposition on $(X\otimes\mathbb{C})^{\mathsf{sch}}$ now gives the following

Lemma *For any pointed projective manifold* (X, x) *and any* $i > 1$ *the subset*

$$\mathsf{Supp}((X \otimes \mathbb{C})^{\mathsf{sch}}, x)$$

is invariant under the $\mathbb{C}^{\times}$ *action on* R*.*

Note: In this geometric case the identification of

$$R(\pi_1((X \otimes \mathbb{C})^{\mathsf{sch}}, x)) \cong R(\pi_1(X, x)),$$

allows us to view $R(\pi_1((X\otimes\mathbb{C})^{\mathsf{sch}}, x))$ as a geometric object as well. Explicitly if $n > 0$ the component R_n of $R(\pi_1(X, x)$ consisting of simple representations of rank n is in a natural way an algebraic variety. Indeed, we can identify R_n with the geometric quotient $\mathsf{Hom}(\pi_1(X), GL_n(\mathbb{C}))^s/GL_n(\mathbb{C})$.

As a consequence we get the following

Theorem (-, Pantev, Toen) *Let* (X, x) *be a pointed projective manifold, then:*

- *If* $\rho \in \mathrm{Supp}((X \otimes \mathbb{C})^{\mathrm{sch}}, x)$ *is an isolated point (for the natural topology on* $R(\pi_1(X, x))$*), then the local system on* X *corresponding to* ρ *underlies a polarizable complex variation of Hodge structures.*
- *If* $\pi_i((X \otimes \mathbb{C})^{\mathrm{sch}}, x)$ *is an affine group scheme of finite type, then each simple factor of the semi-simplification of the* $\pi_1(X, x)$*-module* $\pi_i((X \otimes \mathbb{C})^{\mathrm{sch}}, x)$ *underlies a polarizable* $\mathbb{C}$ VHS.
- *Suppose that* $\pi_1(X, x)$ *is abelian. Then each isolated character*

$$\chi \in \mathrm{Supp}((X \otimes \mathbb{C})^{\mathrm{sch}}, x)$$

must be unitary.

Warning: The support invariants

$$\mathrm{Supp}((X \otimes \mathbb{C})^{\mathrm{sch}}, x)$$

are related to the action of $\pi_1(X, x)$ on $\pi_i(X, x) \otimes \mathbb{C}$ in a highly non-trivial way, which at the moment can be understood only in very special cases.

Nevertheless, the previous theorem can be used to produce explicit new examples of homotopy types which are not realizable by smooth projective varieties.

In order to construct these examples we will need to compute the support invariants explicitly, at least in some cases.

Definition A finitely generated group Γ is called *algebraically good* (relative to $\mathbb{C}$) if the natural morphism of pointed stacks

$$(K(\Gamma, 1) \otimes \mathbb{C})^{\mathrm{sch}} \to K(\Gamma^{\mathrm{alg}}, 1)$$

is an isomorphism.

Remark: Γ is algebraically good iff for any finite dimensional complex representation V of Γ, the natural map $\Gamma \to \Gamma^{\mathsf{alg}}$ induces an isomorphism

$$HH^{\bullet}(\Gamma^{\mathsf{alg}}, V) \cong H^{\bullet}(\Gamma, V).$$

Here $HH^{\bullet}(\Gamma^{\mathsf{alg}}, V)$ denotes the Hochschild cohomology of the affine group scheme Γ^{alg}.

Examples: Finite groups, free groups of finite type, finitely generated abelian groups and the fundamental groups of compact Riemann surfaces are algebraically good groups.

We now have the following:

Theorem (-,Pantev,Toen) *Let $n > 1$ and let (Y, y) be a pointed connected homotopy type so that $\pi_1(Y, y) =: \Gamma$ is algebraically good, $\pi_i(Y, y)$ are finitely generated for $1 < i \leq n$ and $\pi_i(Y, y) = 0$ for $i > n$. Let $\rho : \Gamma \to GL_m(\mathbb{Z})$, and let $\rho_1, \ldots, \rho_r$ be the simple factors of the semi-simplification of $\rho_{\mathbb{C}}$.*

Let $Z = K(\Gamma, \mathbb{Z}^m, n) \times_{K(\Gamma,1)} Y$. If there is a $X \in \mathcal{P}$ so that $\tau_{\leq n} X \cong \tau_{\leq n} Z$, then the real Zariski closure of the image of each ρ_j is a group of Hodge type.

Examples:

(a) Let $\Gamma = \mathbb{Z}^{2g}$ and let ρ be any integral reductive representation, such that $\rho_{\mathbb{C}}$ is not unitary. Then at least one of the characters ρ_j is not unitary and so the real Zariski closure of the image of ρ_j is not of Hodge type.

(b) Let Γ be the fundamental group of a compact Riemann surface of genus $g > 2$ and let $m > 2$. Let $\rho : \Gamma \to SL_m(\mathbb{C})$ be a surjective homomorphism. Then the real Zariski closure of the image of ρ is $SL_m(\mathbb{R})$, which is not a group of Hodge type.

5. Extensions and generalizations

The Hodge decomposition we constructed on $(X \otimes \mathbb{C})^{\mathsf{sch}}$ is only a part of a schematic mixed Hodge structure which includes a *weight filtration*:

Theorem (-, Pantev, Toen) *For any pointed projective manifold (X, x) there exists a natural $\mathbb{C}^{\times\delta}$-equivariant tower of pointed schematic homotopy types*

$$(X \otimes \mathbb{C})^{\mathsf{sch}} \to \ldots \to \mathbb{L}W^{(1)}(X \otimes \mathbb{C}) \to \mathbb{L}W^{(0)}(X \otimes \mathbb{C}),$$

satisfying:

- *There is a natural isomorphism of $\mathbb{C}^{\times\delta}$-equivariant stacks*

$$\mathbb{L}W^{(0)}(X \otimes \mathbb{C}) = B\pi_1(X, x)^{\mathsf{red}}.$$

- *The homotopy fiber $\mathsf{Gr}_W^{(m)}(X \otimes \mathbb{C})$ of the morphism*

$$\mathbb{L}W^{(m)}(X \otimes \mathbb{C}) \to \mathbb{L}W^{(m-1)}(X \otimes \mathbb{C})$$

is representable by a cosimplicial abelian unipotent group scheme.

- *The action of $\mathbb{C}^{\times}$ on the stacks $\mathsf{Gr}_W^{(m)}(X \otimes \mathbb{C})$ satisfies a purity condition.*

The relationship between X and $(X \otimes \mathbb{C})^{\mathsf{sch})}$ is quite difficult to understand. This is related to the existence of groups which are not algebraically good. If X is a complex algebraic manifold, then every point has a Zariski neighborhood whose underlying homotopy type is a $K(\pi, 1)$, where π is a group built by successive extensions of free groups of finite type. Such π's are algebraically good (Serre had shown that these groups are pro-finitely good). Thus the schematization is relatively easy to understand locally in the Zariski topology. This justifies schematic Van Kampen and Lefschetz hyperplane section theorem.

Observe that the schematization functor is left adjoint and thus commutes with homotopy colimits. Therefore one expects that for any open hypecover U_* of X we will have an equivalence

$$(X \otimes \mathbb{C})^{\mathsf{sch}} \cong \mathsf{hocolim}_{[n] \in \Delta} (U_n \otimes \mathbb{C})^{\mathsf{sch}}.$$

Modulo (hard) technical details this result makes $(X \otimes \mathbb{C})^{\mathsf{sch}}$ computable. Of course $(X \otimes \mathbb{C})^{\mathsf{sch}}$

has some shortcomings - it does not behave well under base change and in families - a different more geometric approch to NMHS can be found in math.AG/0006213, Nonabelian mixed Hodge structures, L. Katzarkov, T. Pantev, C. Simpson.

We also expect that the Whitehead products will allow us to obtain new restriction on the rational homotopy types of smooth projective varieties.

One can also look, building on ideas of Deligne, at different realizations of $X \otimes \mathbb{C})^{\text{sch}}$ etale, DR , crystaline and get some arithmetics information about X.