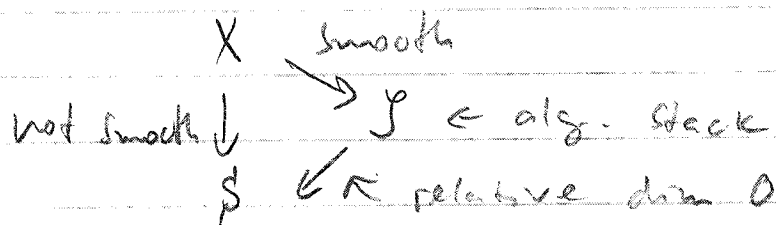


Logarithmic geometry and algebraic stacks

The connection between log geometry and algebraic stacks was first studied by Illusie (following a suggestion of Lafforgue). This approach uses toric stacks i.e. the quotients of a toric varieties by their tori. The approach we will be interested in will be different.

Basic idea: Many interesting morphisms fit naturally into a commutative diagram



- Outline:
- (1) Review of log geometry
  - (2) Define  $\text{Log } \mathcal{Y}$  and explain the relation to toric stacks
  - (3) Log smoothness
  - (4)  $\mathcal{M}_g$ .

## 1. Log geometry

Def: A log structure on a scheme  $X$  is a pair

$$(M, \alpha)$$

consisting of

- $M$  - a sheaf of commutative monoids on  $X_{\text{ét}}$

- Homomorphism of monoids

$$\alpha: M \rightarrow (\mathcal{O}_X, \cdot)$$

Such that

$$\alpha^{-1}(\mathcal{O}_X^\times) \cong \mathcal{O}_X^\times$$

Example: let  $X$  be an integral scheme  
 $U \in X$  - open

Consider  $M_U := \{f \in \mathcal{O}_U \mid f|_D \text{ - unit}\}$   
 then

$$M_U \hookrightarrow \mathcal{O}_U$$

is a log structure on  $U$ .

The log scheme  $(X, M_X \hookrightarrow \mathcal{O}_X)$  should be thought of as the open scheme  $U$  + the way  $U$  sits inside  $X$ .

Remark: If we have any monoid  $M$  + a map  $M \rightarrow \mathcal{O}_X$  (not necessarily satisfying the condition on units)  $\Rightarrow$  we can promote  $M$  to a log structure by taking

$$\mathcal{N} = \mathcal{O}_X \oplus M / \sim \longrightarrow \mathcal{O}_X$$

This construction gives many other examples, e.g.

Example: let  $P$  be any finitely generated monoid and let

$$X = \text{Spec}(\mathbb{Z}[P])$$

Then  $P \rightarrow \mathcal{O}_X$  generates (in the sense of the remark above) a log structure  $M_X \rightarrow \mathcal{O}_X$ .

Def: The category of log structures on  $X$  is the category with

$$\text{Ob} = (X, \mathcal{M}, d) \text{ - log scheme}$$

$$\text{Mor} = \begin{array}{ccc} \mathcal{M} & \xrightarrow{\quad} & \mathcal{M}' \\ & \searrow \alpha & \swarrow \alpha' \\ & \mathcal{O}_X & \end{array} \quad \begin{array}{l} \text{morphism} \\ \text{of monoids.} \end{array}$$

Example: If  $f: X \rightarrow Y$  is a morphism of schemes and if  $\mathcal{M}_Y$  is a log structure on  $Y$   
 $\Rightarrow$  we can define a log structure

$f^* \mathcal{M}_Y$  on  $X$   
 as the log structure corresponding to the monoid  $f^{-1} \mathcal{M}_Y$ , i.e.

$$\begin{array}{ccccc} f^{-1} \mathcal{M}_Y & \longrightarrow & f^{-1} \mathcal{O}_Y & \longrightarrow & \mathcal{O}_X \\ & \searrow & & \swarrow & \\ & & f^* \mathcal{M}_Y & & \end{array}$$

Def: A morphism of log schemes is a pair

$$(X, \mathcal{M}_X) \xrightarrow{(f, f^b)} (Y, \mathcal{M}_Y)$$

s.t.

- $f: X \rightarrow Y$  is a morphism of schemes
- $f^b: f^* \mathcal{M}_Y \rightarrow \mathcal{M}_X$  - morphism of log structures on  $X$ .

Def: If  $X$  is a log scheme, then  $X$  is fine if etale locally on  $X$  there exists a morphism

$$X \xrightarrow{(f, f^b)} (\text{Spec } \mathbb{Z}[I], \text{natural log structure})$$

for some  $I$  -  $\mathbb{Z}$ -generated, so that  $f^b$  is an isomorphism.

Remark: •  $(\text{Spec } \mathbb{Z}[I], \text{nat log str.})$  as above is called a chart for  $X$ .

• From now on all log structures we will look at will be fine.

## The stack $\text{Log } \mathcal{S}$

let  $\mathcal{S}$  be a fixed fine log scheme

let  $\underline{\mathcal{S}}$  be the underlying scheme

We consider a fibered category

$$\text{Log } \mathcal{S} \rightarrow (\underline{\mathcal{S}}\text{-schemes})$$

$$X/\mathcal{S} \rightarrow \underline{X}/\underline{\mathcal{S}}$$

where

$$\text{Ob}(\text{Log } \mathcal{S}) = X \rightarrow \mathcal{S} \text{ - morphism of fine log schemes}$$

$$\text{Mor}(\text{Log } \mathcal{S}) : \begin{array}{c} X' \xrightarrow{(f, f^b)} X \\ \downarrow \swarrow \\ \mathcal{S} \end{array}$$

- morphisms of log schemes  
s.t.  $f^b$  is an isomorphism.

Example: If we take  $\mathcal{S} = (\text{Spec } \mathbb{Z}, \mathcal{O}^*)$   
 - the initial object in the category  
 of log schemes, then if we put

$$\text{Log} := \text{Log } \mathcal{S}$$

$\Rightarrow \text{Log}(\mathbb{T}) =$  groupoid of log structures  
 on  $\mathbb{T}$ .

Theorem (Olsson)  $\text{Log } \mathcal{S}$  is an algebraic  
 stack which is locally of  
 finite presentation on  $\mathcal{S}$ .

Remark:  $\Delta = \text{Log } \mathcal{S} \rightarrow \text{Log } \mathcal{S} \times \text{Log } \mathcal{S}$   
 is not separated but is locally  
 separated.

A presentation of  $\text{Log}$

$\mathbb{P} \mapsto (\text{Spec } \mathbb{Z}[\mathbb{P}], \text{canonical log structure})$

!!

$\mathbb{D}(\mathbb{P})$

Theorem:  $\mathbb{1} \rightarrow \text{Spec}(\mathbb{Z}[P]) \rightarrow \text{Log}$

is a flat cover of  $\text{Log}$  which is smooth if the group  $P^g$  associated to the monoid  $P$  is torsion free.

In fact we can make this more precise.

Consider  $\mathcal{D}(P) := \text{Spec}(\mathbb{Z}[P])$

Then

$\mathcal{D}(P^g)$  acts on  $\mathcal{D}(P)$

since

$$\mathcal{D}(P)(R) = \text{Hom}_{\text{Monoids}}(P, R) \cup$$

$$\mathcal{D}(P^g)(R) = \text{Hom}_{\text{Groups}}(P^g, R^\times)$$

Consider the toric stack

$$\mathcal{Y}_P := [\mathcal{D}(P) / \mathcal{D}(P^g)]$$

then we have



Theorem:  $\mathcal{Y}_P \rightarrow \text{Log}$  is étale.

Remark:  $\mathcal{Y}_P \rightarrow \text{Log}$  is not an isomorphism

Example:  $P = \mathbb{N}^r$   
 $\mathcal{D}(P) = \mathbb{A}^r \rightrightarrows \mathbb{G}_m^r = \mathcal{D}(P)$

In fact we can identify  $\mathcal{Y}_P$  with the classifying stack of pairs  $(M, \beta)$  where

$M$  - log structure on  $\text{Spec } \mathbb{Z}$

$\beta: P \rightarrow M/\mathcal{O}^*$  - morphism of monoids which locally lift to a chart

Example:  $[\mathbb{A}^r / \mathbb{G}_m^r] = \{ (M, \beta) \mid \mathbb{N}^r \xrightarrow{\beta} M/\mathcal{O}^* \}$

Log smoothness

let  $\mathcal{P}$  be a property of a representable

morphism of schemes

Def: If  $f: X \rightarrow S$  is a morphism of log schemes  $\Rightarrow$  we say that  $f$  has property  $\mathcal{P}$  if the natural map

has the property  $\mathcal{P}$ .

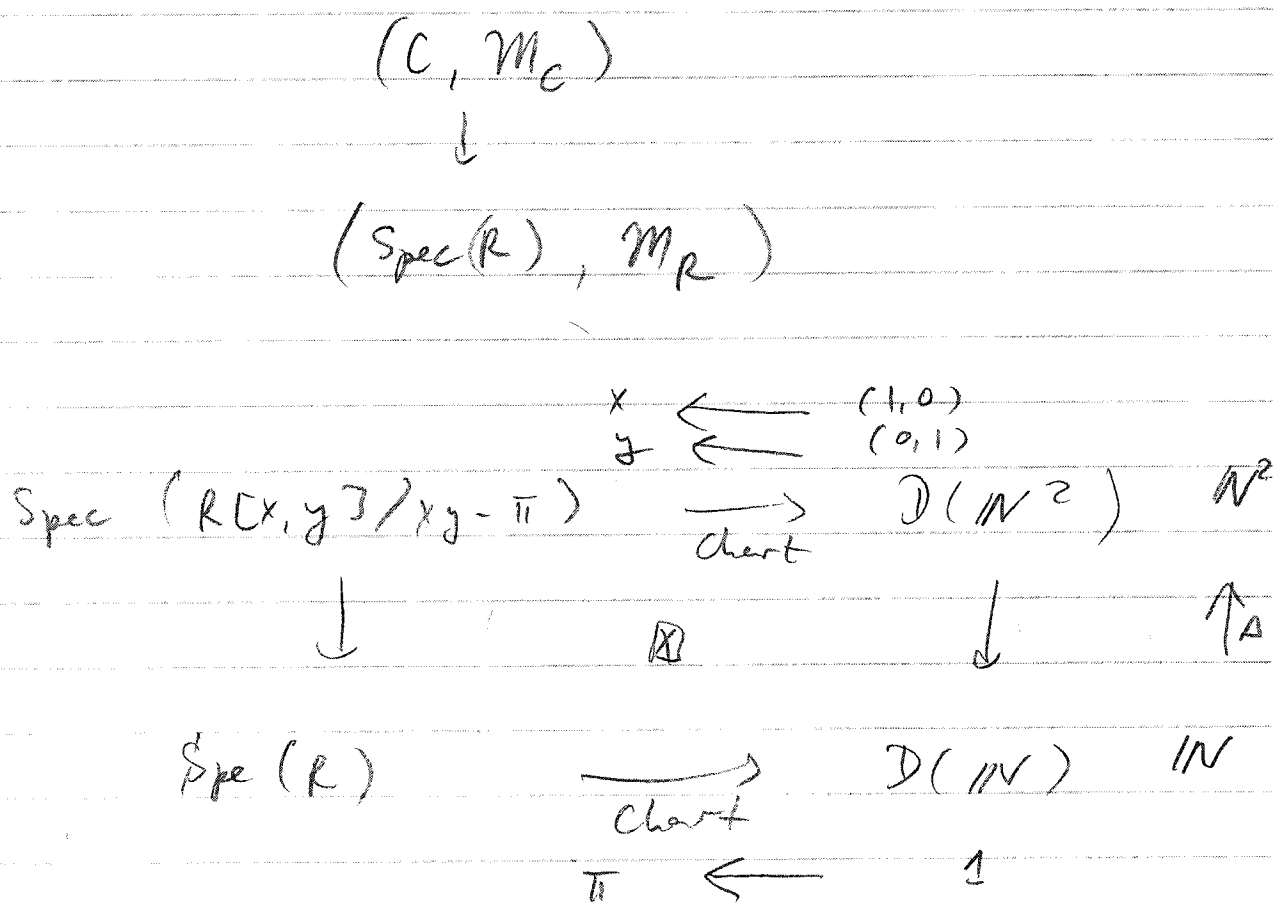
Example: If  $R$  is a DVR with a uniformizing parameter  $\pi$   
 $\Rightarrow$  if  $k = \text{Frac}(R)$  we may look at a semistable curve

$$\begin{array}{ccc} C_k & \subset & C \\ \downarrow & & \downarrow \\ \text{Spec}(k) & \subset & \text{Spec}(R) \end{array}$$

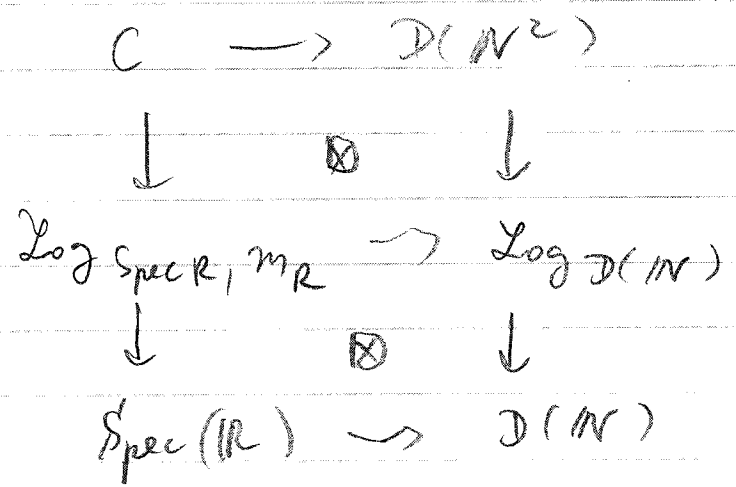
$\Rightarrow$  the natural local map

$$\begin{array}{c} \text{Spec}(R[x, y] / (xy - \pi)) \\ \downarrow \\ \text{Spec } R \end{array}$$

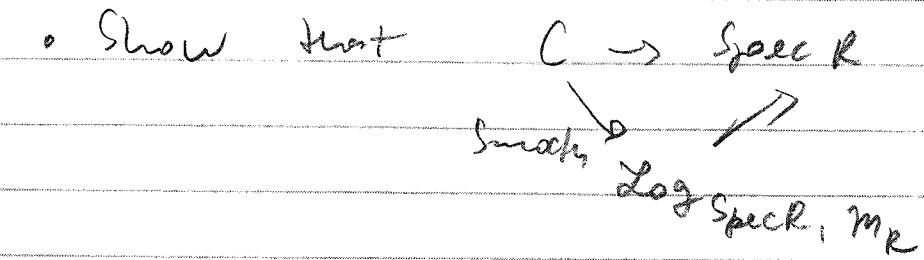
gives rise to a natural morphism of log schemes



In fact this gives



Exercise :



• Show that the relative differentials for

$$C \rightarrow \text{Log Spec } R, m_p$$

are exactly the differentials on  $C$  with logarithmic poles along the closed fiber.

$\overline{\mathcal{M}}_g$

The stack  $(\overline{\mathcal{M}}_g, \mathcal{M}_{\infty})$  is now a log stack (with the log structure defined above)

In particular  $(\overline{\mathcal{M}}_g, \mathcal{M}_{\infty})$  will be a fibered category over the category of log schemes

We will define now a fibered category  $\mathcal{M}_g^{\text{log}}$  over the category of log schemes independently.

Objects / T : morphisms of log schemes  $f: C \rightarrow T$

s.t.

- (1)  $f$  log smooth proper vertical, integral
- (2)  $\forall F \rightarrow T \Rightarrow C_F \rightarrow F$

satisfies

- (i)  $C_F$  - reduced curve of genus  $g$
- (ii)  $C_F \rightarrow \text{Log } F$  is s.t.

If  $(R^1 \mathcal{O}_{C_F / \text{Log } F})^\vee =: \mathcal{O}_{C_F}$

then

$H^0(C_F, \mathcal{O}_{C_F}) = 0$ .

Theorem:  $\mathcal{M}_g^{\text{log}}$  is equivalent to the log stack defined by  $(\mathcal{M}_g, \mathcal{M}_\infty)$ .

The proof involves two steps

Step 1:  $\mathcal{F}$  fibered category  $(\mathcal{F}, M_{\mathcal{F}})$   
Such that

$$M_{\mathcal{F}}^{\text{log}} \cong \text{associated stack}$$

Step 2: the fibered product

$$\begin{array}{ccc}
 \mathcal{F} & \leftarrow & \mathcal{F} \times_{\mathcal{Z}[\mathbb{L}]} \\
 M_{\mathcal{F}} \downarrow & & \downarrow \text{Log} \\
 \text{Log} & \leftarrow & \text{Spec } \mathcal{Z}[\mathbb{L}] \quad P = \mathbb{N}^r
 \end{array}$$

is algebraic

Now note that the defo theory of  $\mathcal{F} \times_{\text{Log}} \mathcal{Z}[\mathbb{L}]$  is easy to understand

Indeed given  $R$  -  $\mathcal{Z}[\mathbb{L}]$  algebra and solid arrows

$$\begin{array}{ccc}
 C & \dashrightarrow & C' \\
 \downarrow & & \downarrow \\
 (\text{Spec } R, m_R) & \hookrightarrow & (\text{Spec } R', m_{R'})
 \end{array}$$

we need to find  $C'$  and the dotted arrows

This can be rewritten for

$$\begin{array}{ccc}
 \underline{C} & \xrightarrow{\quad} & \underline{C}' \\
 \downarrow & \boxtimes & \downarrow \\
 \text{Log}(R, \mathcal{M}_R) & \xrightarrow{\quad} & \text{Log}(R', \mathcal{M}_{R'})
 \end{array}$$

The deformation is given by

$$H^i(\underline{C}, \textcircled{H} \underline{C} / \text{Log}(\text{Spec } R, \mathcal{M}_R))$$