Outline

<u>0.</u> Thank organizers, joint w/ Martin Olsson

reference:

<u>1.</u> State problem: <u>Setup</u>

$\mathfrak{X} \xrightarrow{p} S$, $\mathcal{F}$ locally f. pres'd q-coh. $\mathcal{O}_{\mathfrak{X}}$-module (in small étale site of $\mathfrak{X}$).

DM stack ($\mathfrak{X}$), alg. sp (S)

p locally f. pres'd + sep'd

Define a contravariant functor

$Q = Q(\mathcal{F}/\mathfrak{X}/S) : S\text{-schemes} \longrightarrow \text{Sets}$

where, given $T \xrightarrow{g} S$, form fiber square

$$\begin{tikzcd}
T \times_S \mathfrak{X} \arrow[r, "pr_2"] \arrow[d, "pr_1"'] & \mathfrak{X} \arrow[d, "p"] \\
T \arrow[r, "g"] & S
\end{tikzcd}$$

and $Q(T) =$ the set of quotients $pr_2^* \mathcal{F} \twoheadrightarrow \mathcal{G}$ s.t.

(1) $\mathcal{G}$ is a locally f. pres'd $\mathcal{O}_{T \times_S \mathfrak{X}}$-module

(2) $\mathcal{G}$ is flat over T

(3) the support of $\mathcal{G}$ is proper over T.

<u>Defn:</u> Q = Quot functor of $\mathcal{F}/\mathfrak{X}/S$.

Special case: $\mathcal{F} = \mathcal{O}_{\mathcal{X}}$. Then $\mathcal{G} = \mathcal{O}_{\mathcal{Z}}$ for some closed substack $\mathcal{Z} \subset T \times_S \mathcal{X}$ which is flat and proper over T.

Def'n: $H(\mathcal{X}/S)$ = Hilbert functor defined to be $Q(\mathcal{O}_{\mathcal{X}}/\mathcal{X}/S)$.

Question: Is Q represented by an algebraic space? Are the connected components q-projective? Is there a Hilbert polynomial?

One immediate answer: If $\mathcal{X} = [Y/G]$ with G a finite flat group scheme $/S$.

Y an algebraic space.

$$\begin{tikzcd} G \times_S Y \arrow[r, "pr_2"] \arrow[d, "m"'] & Y \arrow[d, "f"] \\ Y \arrow[r, "f"] & \mathcal{X} \end{tikzcd}$$

, $\phi : m^* \mathcal{F}_Y \to pr_2^* \mathcal{F}_Y$ an isomorphism satisfying cocycle condition.

set $\mathcal{F}_Y = f^* \mathcal{F}$

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Have a natural transformation

$f^*: Q(\mathcal{F}/\mathcal{X}/S) \longrightarrow Q(\mathcal{F}_Y/Y/S)$.

Have two natural transformations

~~$Q(\mathcal{F}_Y/Y/S) \rightarrow Q(G\times_S Y$~~

$Q(\mathcal{F}_Y/Y/S) \rightarrow Q(m^*\mathcal{F}_Y/G\times_S Y/S)$

m^* and $\alpha\circ pr_2^*$: $(\mathcal{F}_Y \overset{\alpha}{\twoheadrightarrow} \mathcal{G}) \mapsto$

$(m^*\mathcal{F}_Y \overset{\phi}{\xrightarrow{\sim}} pr_2^*\mathcal{F}_Y \overset{pr_2^*\alpha}{\twoheadrightarrow} pr_2^*\mathcal{G})$

And by descent

$Q(\mathcal{F}/\mathcal{X}/S) \cong$ ~~$Q(\mathcal{F}_Y/Y/S) \times_{\xrightarrow{(m^*,\alpha\circ pr_2^*)}} Q(m^*\mathcal{F}_Y/G\times_S Y/S)$~~

$$= \underset{\xrightarrow{(m^*,\,\alpha\circ pr_2^*)}}{Q(\mathcal{F}_Y/Y/S)} \times_{Q(m^*\mathcal{F}_Y/G\times_S Y/S)\times Q(m^*\mathcal{F}_Y/G\times_S Y/S)} \underset{\xleftarrow{\Delta}}{Q(m^*\mathcal{F}_Y/G\times_S Y/S)}$$

So reduced to representing Quot for algebraic spaces which was proved by Artin.

General case (1) Artin's approach

(2) Grothendieck's approach

Results:

Thm 1: ~~AA~~ As above^{Qu}, $Q(\mathcal{F}/\mathfrak{X}/S)$ is represented by an algebraic space which is locally f. pres'd and separated over S.

Thm 2: (Assume S = affine scheme) If $\mathfrak{X}$ is a tame, global quotient whose coarse moduli space M is q-proj (resp. proj.). Then Q is represented by a scheme whose connected components are q-proj. (resp. proj.).

Artin's approach: In "Algebraization of Formal Moduli, I", Artin gives axioms for when a functor is represented by an algebraic space. He checks his axioms for the case of $Q(\mathcal{F}/X/S)$ where X is an algebraic space (with one minor mistake).

We verify Artin's axioms. The main step is to prove the $^{\mathrm{am}}$stack version of Grothendieck's existence theorem.

Finite cover theorem of Vi, L-MB and EHKV.

Minor mistake is that Artin's obstruction group isn't sufficient when $\mathcal{F}$ isn't flat over S. [Aside in obstruction theory].

<u>Corollary</u>: Given $\mathcal{F}/\mathcal{X}/S$ ~~with $\widetilde{\mathcal{X}}$~~ such that $\mathcal{F}$ has proper support over S, then the "flattening stratification" of S exists, i.e. a surjective, finitely-pres'd monomorphism $\Sigma \xrightarrow{g} S$ such that for any $T \xrightarrow{h} S$, pullback of $\mathcal{F}$ to $T \times_S \mathcal{X}$ is flat over T iff h factors through g.

<u>Grothendieck's approach</u> (as improved on by Mumford + Sernesi).

(1) Define Hilbert polynomial P and show $Q = \coprod Q^P$.

(2) Prove boundedness of Q^P

(3) Use boundedness to get a monomorphism

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$Q^P \hookrightarrow$ Grassmannian, $\mathbb{G}$

(4) Perform flattening stratification of an appropriate sheaf over $\mathbb{G}$ to prove $Q^P \hookrightarrow \mathbb{G}$ is a closed immersion.

We do roughly the same thing.

<u>Hilbert polynomial</u>: Given any $\operatorname{Spec} k \xrightarrow{s} S$ and a l. f. presd sheaf $\mathcal{G}$ on $\operatorname{Spec} k \times_S \mathcal{X}$ which has proper support, then $\chi(\mathcal{X}_k, \mathcal{G})$ is finite.

Consider the function

$$P : K^0(\mathcal{X}) \longrightarrow \mathbb{Z}, \quad P([\mathcal{E}]) = \chi(\mathcal{X}_k, \mathcal{G} \otimes pr_2^* \mathcal{E}).$$

This is a well-defined group homomorphism which we call the Hilbert polynomial.

<u>Fact</u>: Given a conn'd scheme T, $T \xrightarrow{s} S$ and $\mathcal{G}$ a loc. f. presd sheaf on $T \times_S \mathcal{X}$

flat and proper over T, then the Hilbert poly. map $T \to \mathrm{Hom}(K^0(\mathcal{X}), \mathbb{Z})$ is const.t.

So $Q = \coprod_{P} Q^P$.

Let $f : \mathcal{X} \to M$ denote the coarse moduli space. Assume $\mathcal{X}$ is *tame*, i.e. for any geometric point p of $\mathcal{X}$, its stabilizer group has order prime to char $k(p)$,

Abramovich -Vistoli "Compactifying the space of stable maps."

Then (1) formation of M is compatible with base-change on S,

(2) f_* preserves quasi-coherent / coherent sheaves and is exact (and compatible with arbitrary base-change)

(3) f_* preserves "flatness over S".

If $\mathcal{E}$ is a locally free sheaf on $\mathcal{X}$, then there is a natural transformation

$$Q^{p}(\mathcal{F}/\mathcal{X}/S) \longrightarrow Q^{p'}(f_* \mathcal{H}om(\mathcal{E},\mathcal{F})/M/S)$$

by $(pr_2^*\mathcal{F} \twoheadrightarrow \mathcal{G}) \mapsto$

$$(f_* \mathcal{H}om(pr_2^*\mathcal{E}, pr_2^*\mathcal{F}) \twoheadrightarrow f_*\mathcal{H}om(pr_2^*\mathcal{E},\mathcal{G})).$$

If we assume that M is a q-proj. S-sch., then $Q^{p'}$ is a q-proj. S-scheme. How to find $\mathcal{E}$ s.t. $Q^{p} \to Q^{p'}$ is a monomorphism?

Consider $f^* f_* \mathcal{H}om(\mathcal{E},\mathcal{F}) \otimes \mathcal{E} \to \mathcal{F}$. Suppose this is surjective for all $\mathcal{F}$. Then if $\mathcal{K} =$ kernel $(f_*\mathcal{H}om(pr_2^*\mathcal{E},pr_2^*\mathcal{F}) \to (\text{''}, \mathcal{G}))$ then $\mathcal{G}$ = cokernel of composition $f^*\mathcal{K}\otimes\mathcal{E} \to f^* f_*\mathcal{H}om(pr_2^*\mathcal{E},pr_2^*\mathcal{F})\otimes\mathcal{E} \to \mathcal{F}$. So we can recover a point in Q^{p} from its image in $Q^{p'}$.

By a theorem of Edidin/Hassett/Kresch/Vistoli, this condition on $\mathcal{E}/\mathcal{F}$ (granting M q-proj.) is exactly equivalent to asking that $\mathcal{X}$ is a global quotient.

Finally, use flattening stratification to prove $Q^{P} \hookrightarrow Q^{P'}$ is, in fact, a closed immersion.

Stacky surfaces?