1

Riemann-Roch for Quotients & Todd classes of toric varieties, MSRI March 2002

GRR says If Z is a smooth scheme then the map

$$\tau_Z : K_0(Z) \to CH^*(Z)_{\mathbb{Q}} \quad \text{is covariant for proper morphisms}$$

$$[\mathcal{E}] \to ch(\mathcal{E})\, Td(T_X)$$

when Z is singular BFM extended this show that there is an iso

$$\tau_Z\ K_0(Z) \to CH^*(Z) \quad (\text{where } CH^* Z \text{ are Chow gps indexed by codim})$$

st.
1) τ_Z is covariant for proper morphisms
2) If $\mathcal{E}$ is a vb then $\tau([\mathcal{E}]) = ch(E)\, \tau(\mathcal{O}_X)$

<u>Problem:</u> Given an explicit formula for τ_Z where $Z = X/G$
~~and X is smooth alg space & G a linearly~~
when G acts properly on a smooth alg space X (Here G is a lin alg gp)
This is an alg version of Kawasaki-RR Theorem in Diff Geo

Some basic facts.

1) ~~There~~ is The map $inv : K_0^G X \to K_0(X/G)$ is covariant for proper morphisms
$[\mathcal{E}] \to [\mathcal{E}^G]$ where $\mathcal{E}^G$ is subsheaf of invariant sections of $\mathcal{E}$
(this is the pushforward of the proper non-rep map of stacks $[X/G] \to X/G$)

2). There is an iso $\Phi : CH_G^* X_{\mathbb{Q}} \to CH^*(X/G)_{\mathbb{Q}}$ covariant for proper equiv morphisms $Y \to X$

3) Map $K_0^G X \xrightarrow{ch^G Td^G(X)} CH_G^* X$ is covariant for proper equiv morphisms

Putting these facts together gives a diagram where all arrows are covariant for proper morphisms

$$\begin{tikzcd}
K_0^G X \arrow[r, "ch^G\, Td^G(X)"] \arrow[d, "inv"'] & (CH_G^* X)_{\mathbb{Q}} \arrow[d, "\Phi"] \\
K_0(X/G) \arrow[r, "\tau"] & CH^*(X/G)_{\mathbb{Q}}
\end{tikzcd}$$

Eg $G = \mathbb{C}^*$ acting with weights $(1,2)$
on $X = \mathbb{A}^2 - \{0\}$, $X/G = \mathbb{P}^1$
Let $\mathbb{1}$ be the ~~natural~~ trivial 1-dim rep of $\mathbb{C}^*$
then $\tau((\mathcal{O}_X \otimes \mathbb{1})^G) = 1 + [P]$
while $\Phi(ch^G(\mathbb{1})\, Td^G(X))$
$= 1 + 3/4\,[P]$

~~Naive hope: Diagram commutes~~

Unfortunately diagram DOES NOT COMMUTE

~~(If $X = \mathbb{A}^2 - \{0\}$, $G = \mathbb{C}^*$ acting with weights $(1,2)$ then $\Phi(T$~~

Two approaches to dealing with this.

1. Toen's approach define $CH^*_{stack}([X/G])$
   and ~~show~~ define $K_0([X/G]) \xrightarrow{ch^\circ Td^\circ} CH^*_{stack}([X/G])$
   which is covariant for proper smooth morphisms of stacks. This allows you to compute Euler characteristics though not quite what we want

2. Equivariant approach view $K_0^G X$ as an $R(G)_{\mathbb{Q}}$ module
   G acts with finite stabilizers $\Rightarrow$ $K_0^G X$ is supported at a finite # of points of spec $R(G)$.

spec $R(G)$ has a distinguished point 1 corresponding to $\ker: R(G) \xrightarrow{\text{rank}} \mathbb{Q}$

~~Some facts~~.

<u>Prop:</u> If $\alpha \in (K_0^G X)$, then $\tau(\text{inv}(\alpha)) = \Phi(ch^G(\alpha)\, Td^G(T_X))$.

This follows from fact that $\exists X' \to X$ finite on which G acts freely and ~~construction~~ G-equivariant generalization of BFM to ~~[illegible]~~ by E-G.

Basic problem then is to compute $\cancel{\tau(E_\sigma)}$ $\tau(\text{inv}(\alpha_\sigma))$ where α_σ is supported at $\sigma \in$ spec $R(G)$.

In this talk I'll show what to do when G is abelian (GL_n can be dealt with via a max torus) & for simplicity we'll work over $\mathbb{C}$.

Then $G \simeq$ spec $R(G) \otimes \mathbb{C}$ & σ can be thought of as an elt of $\mathbb{C}$.

Consider $X^\sigma = \{x \in X \mid \sigma x = x\}$, $\langle\sigma\rangle$ acts trivially on it

$$K_0^G(X^\sigma) = K^{G/\langle\sigma\rangle}(X) \otimes_{R(G/\sigma)} R(G)$$

So if $\beta \in K_0^G(X^\sigma)$ ~~set $\beta($~~ $\beta = \sum \bar{\beta}_i \otimes x_i$

Set $\beta(\sigma) = \sum \bar{\beta}_i \otimes x_i(\sigma)\, x_i$

$\beta(\sigma)$

Prop: If σ acts trivially on $\cancel{X}$ Y & $\beta \in K_0^G Y$.
then $\operatorname{inv}(\beta_{\sigma}) = \operatorname{inv}\beta(\sigma)$

Using proposition & localization for torus actions we obtain

Thm: G acts properly on smooth space X
$\mathcal{E}$ an equiv vb on X then $\tau(\mathcal{E}^G) = \int\Big(\sum_{\sigma \in \operatorname{Supp} K^G(X)} \iota_{\sigma *} ch^G\Big(\frac{\iota_\sigma^* \mathcal{E}(\sigma)}{\lambda_{-1}(N_\sigma)}\Big) Td^G(X^\sigma)\Big)$

Where $\iota_\sigma : X^\sigma \to X$ is the inclusion of the fixed locus
(Note X^σ need not be connected but it is smooth.)

Application to toric varieties. Can use to compute $\tau(\mathcal{O}_X)$ where X is a simplicial toric variety. We rederive B-V's formula using global coordinates for toric varieties

Notation Fix a d-dim torus T. $N = \operatorname{Hom}(\mathbb{G}_m, T)$ & $M = \operatorname{Hom}(T, \mathbb{G}_m)$
a T-toric variety X is determined by its fan Σ in $N \otimes \mathbb{R}$
Σ simplicial $\Leftrightarrow$ X has finite ab. quotient sings
(ie every cone gen by indep vectors)

For each cone σ in Σ there is a codim σ subvariety V_σ of X. ~~A~~
($V_\sigma = X_\Sigma^{T_\sigma}$ where $T_\sigma \subset T$ is subgp with gp of 1ps subgps $\sigma \cap N \subset N$)

Global coordinates: let G be the kernel of ~~Define a~~ map $(\mathbb{C}^*)^{\Sigma(1)} \longrightarrow T$ ← 1 dim cones in Σ

$(t_\tau)_{\tau \in \Sigma(1)} \to \prod_{\tau \in \Sigma(1)} n_\tau(t)$ where $n_\tau : \mathbb{C}^* \to T$ is the 1-ps subgp det by τ.

~~Let G be~~ p
Let $Z(\Sigma) \subset \mathbb{A}^{\Sigma(1)}$ be the union of linear subspaces defined by ~~equ~~ ideal $B(\Sigma) = \langle \prod_{\tau \notin \sigma(1)} x_\tau \mid \sigma \in \Sigma \rangle$
(Here x_τ's are coordinates on $\mathbb{A}^{\Sigma(1)}$)

Let $W = \mathbb{A}^{\Sigma(1)} - B(\Sigma)$

Theorem (Cox) G acts properly on W & $W/G \cong X$

Under quotient map ~~coordinate hyperplanes~~ the image of the linear subspace $\underset{W_\sigma}{V(\langle x_{\tau_i} \mid \tau_i \in \sigma(1)\rangle)}$ is $V(\sigma)$.

Let $G_\sigma \subset G$ be the stabilizer of W_σ & $G_\Sigma = \bigcup G_\sigma$ a finite ~~subgp~~ subset of G

Finally let a_τ be ~~proj~~ the character of T defined by projection to first factor

Applying our formula we obtain

$$\tau(\mathcal{O}_X) = \tau(\mathbb{1}^G) = \Phi\Big(\sum_{g \in G_\Sigma} \prod_{\tau \in \Sigma(1)} \frac{x_\tau}{1 - a_\tau(g) e^{-x_\tau}}\Big)$$

$$= \sum_{g \in G_\Sigma} \prod_{\tau \in \Sigma(1)} \frac{[V_\tau]}{1 - a_\tau(g) e^{-[V_\tau]}}\Big) \in \cancel{A}H^*(X)$$

Remark: B-V get an analogous formula in T-equivariant coh (or chow) ring. This follows from an equiv version of our formula too.